QUARTERLY REPORT

PROJECT B-208

"STUDY OF THE PRESSURE DISTRIBUTION ON OSCILLATING PANELS IN LOW SUPERSONIC FLOW WITH TURBULENT BOUNDARY LAYER"

Contract NAS2-2897

11 June 1965 to 10 September 1965

Prepared for
Ames Research Center
National Aeronautics and Space Administration
Moffett Field, California

1965

Engineering Experiment Station
GEORGIA INSTITUTE OF TECHNOLOGY
Atlanta, Georgia

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PREFACE

This report covers research initiated by the National Aeronautics and Space Administration, Ames Research Center, Moffett Field, California, and performed under Contract NAS2-2897. The work is administered by Mr. P. A. Gaspers.

The principal investigator of the program is Dr. E. F. E. Zeydel. Professor A. C. Bruce made a major contribution in this report period.

This report covers work done under this contract for the period June 11, 1965, to September 11, 1965.

LIST OF SYMBOLS

Variables:

t	=	time
x	=	plate coordinate in stream direction
y	=	coordinate normal to plate
u,v	=	x and y velocity components, respectively
p	=	static pressure
p_x, p_y, p_z	=	normal stresses
τ_{ij}	=	stress in j^{th} direction on i^{th} plane
ρ	=	fluid density
U	=	local potential flow velocity
T	=	absolute temperature
μ	=	coefficient of viscosity
δ	=	boundary layer thickness

Superscripts and other:

t	=	total, sum of "mean" and fluctuation
$'$	=	fluctuation component or differentiation with respect to x
$-$	=	mean component

INTRODUCTION

The calculation of the unsteady pressure distribution on finite, oscillating panels exposed to a turbulent boundary layer in low supersonic flow has not been conclusively established. Inasmuch as this problem is suggested by discrepancies between experiment and theory pertaining to panel flutter wherein the theory is based on a potential flow description, it would seem that a reasonable explanation may be found in an investigation of the viscous or boundary layer effects present.

In general, the mechanisms of viscosity, turbulence, unsteadiness, and compressibility all enter the problem. To a limited extent, some of the effects of these mechanisms may be incorporated into potential flow descriptions; however, each mechanism will, in general, also manifest itself in the boundary layer. For purposes of the present discussion we adopt the viewpoint that the effects of regular unsteadiness, due to perturbations created by a sinusoidally oscillating wall, are the primary effects to be investigated. Unfortunately, consideration of the appropriate ranges of the dimensionless variables that this unsteadiness enters into does not serve to simplify the problem.

As a consequence then of physical observations and the present state of the knowledge of turbulent boundary layers it seems fair to assume that the problem, as a whole, is complicated in its representation by all of these mechanisms and not obviously amendable to simplification. In addition, one requirement of this analysis is to investigate the boundary layer thickness range from zero to ten per cent of the panel length. Inasmuch as the panel oscillations have a small amplitude to wave length ratio and the panel length may be considerably larger than the wave length, the boundary layer is not thin in every case under consideration. In other words, the usual boundary layer order of magnitude analysis may not be strictly valid and, if not, an additional effect of unsteadiness due to panel shape is incurred. A logical approach then is to investigate the problem in such a way that the relative importance of these different effects may be examined wherever possible. To this end, a very general formulation and description of the problem is deemed necessary to examine it in a proper, if somewhat complicated, perspective.

In view of the fact that the boundary layer is turbulent and the flow is generally complicated, the most tractable general formulation of the problem requires that consideration be restricted to the "mean" turbulent flow in two dimensions. The resulting equations of motion pertaining to the regular unsteadiness induced by the wall are the "mean" Navier-Stokes equations together with the unsteady, compressible continuity equation for the mean flow. These equations do not permit exact solution; as a consequence, an approximate boundary layer type of approach may be employed to retain average effects of the boundary layer. This approach, in the case of a thin boundary layer, involves consideration of one integral equation obtained from the boundary layer equations and describing the relation between skin friction, pressure gradient, and boundary layer displacement and momentum thicknesses. In the present case of a generally thick boundary layer there are two resultant integral equations, one similar to but more complicated than the usual thin boundary layer equation and a second integral equation describing the change in pressure or pressure gradient across the boundary layer. It is the initial purpose of this analysis to formulate these equations in convenient forms and discuss the possibilities of their application to the present problem.

MEAN FLOW EQUATIONS

Knowledge of the pressure distribution on a sinusoidally oscillating plate exposed to a thick turbulent boundary layer must be determined from the governing differential equations of continuity, momentum, and energy together with an equation of state. In a first attempt at simplification of this complex problem, the flow will be assumed to be two-dimensional and energy considerations will be avoided by assuming the flow to be isothermal. The applicable governing equations of motion and continuity are then:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial p_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \tag{2}$$

$$\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = \frac{\partial p_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} \tag{3}$$

together with the equation of state:

$$p = \rho RT \tag{4}$$

where the following definitions are employed:

$$p = -\frac{1}{3}(p_x + p_y + p_z)$$

$$p + p_x = -\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu \frac{\partial u}{\partial x} \tag{5}$$

(continued next page)

$$p + p_y = -\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y}$$

$$p + p_z = -\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{5}$$

Alternately, using the continuity condition, equation (1), the momentum equations (2) and (3) may be conveniently written as:

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) = \frac{\partial p_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \tag{6}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2) = \frac{\partial p_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} \tag{7}$$

The turbulent flow may then be represented as composed of "mean" and fluctuating components in all dependent variables. It is understood here that the fluctuating components are random in nature and not to be confused with the regular fluctuations due to wall oscillations, which are by definition part of the "mean" flow. The fluctuations due to turbulence are then defined by the primed quantities as follows:

$$u = \bar{u} + u' \;, \;\ldots$$

$$\rho u = \overline{\rho u} + (\rho u)' \;, \;\ldots$$

$$\rho = \bar{\rho} + \rho'$$

$$p_i = \bar{p}_i + p_i' \quad ; \quad i = x,y,z$$

$$\tau_{ij} = \bar{\tau}_{ij} + \tau_{ij}' \;; \quad i,j = x,y,z \quad ; \quad i \neq j \tag{8}$$

These definitions are substituted into equations (1), (6), and (7) and the resulting equations are time averaged over a period large in comparison to the "time of random fluctuations" but small in comparison to the "time of mean variations," for example, the period of regular disturbances due to wall oscillations. The resulting "mean" equations are as given by Van Driest (Ref. 1).

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x}(\overline{\rho u}) + \frac{\partial}{\partial y}(\overline{\rho v}) = 0 \tag{9}$$

$$\bar{\rho}\frac{\partial \bar{u}}{\partial t} + \overline{\rho u}\frac{\partial \bar{u}}{\partial x} + \overline{\rho v}\frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial t}(-\overline{\rho' u'})$$

$$+ \frac{\partial}{\partial x}\left[\bar{p}_x - \overline{(\rho u)' u'}\right] + \frac{\partial}{\partial y}\left[\bar{\tau}_{yx} - \overline{(\rho v)' u'}\right] \tag{10}$$

$$\bar{\rho}\frac{\partial \bar{v}}{\partial t} + \overline{\rho u}\frac{\partial \bar{v}}{\partial x} + \overline{\rho v}\frac{\partial \bar{v}}{\partial y} = \frac{\partial}{\partial t}(-\overline{\rho' v'})$$

$$+ \frac{\partial}{\partial y}\left[\bar{p}_y - \overline{(\rho v)' v'}\right] + \frac{\partial}{\partial x}\left[\bar{\tau}_{xy} - \overline{(\rho u)' v'}\right] \tag{11}$$

where the barred quantities represent the "mean" values in the conventional manner (Ref. 2). If, in addition to the shear stress as defined by equation (6), the normal stresses are conventionally defined, then one obtains:

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\tau_{yy} = 2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

and

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{12}$$

These definitions allow equations (10) and (11) to be written in terms of the mean pressure, $\bar{p}$, by use of

$$\bar{p}_x = -\bar{p} + \bar{\tau}_{xx}$$

$$\bar{p}_y = -\bar{p} + \bar{\tau}_{yy} \tag{13}$$

It is to be noted that the "mean" values of the products of fluctuating quantities enter into equations (10) and (11) in the form of two distinct alternations when the resulting equations are compared to the analogous laminar flow equations. The first alternation is the appearance of four apparent stresses due to the "mean" of products of the fluctuating quantities $\overline{(\rho u)'u'}$, $\overline{(\rho v)'u'}$, $\overline{(\rho v)'v'}$, and $\overline{(\rho u)'v'}$; the second alternation is due to the unsteadiness of the "mean" products $\overline{\rho'u'}$ and $\overline{\rho'v'}$.

This first alternation is handled in a more compact form by defining the "total" stresses.

$$\bar{\tau}_{xx} - \overline{(\rho u)'u'} \equiv \bar{\tau}_{xx}^{\ t}$$

$$\bar{\tau}_{yy} - \overline{(\rho v)'v'} \equiv \bar{\tau}_{yy}^{\ t}$$

$$\bar{\tau}_{yx} - \overline{(\rho v)'u'} \equiv \bar{\tau}_{yx}^{\ t}$$

$$\bar{\tau}_{xy} - \overline{(\rho u)'v'} \equiv \bar{\tau}_{xy}^{\ t} \neq \bar{\tau}_{yx}^{\ t} \tag{14}$$

The second alternation is, in effect, hidden by use of the identity

$$\frac{\partial}{\partial t}(\overline{\rho u_i}) = \frac{\partial}{\partial t}(\bar{\rho}\bar{u}_i) + \frac{\partial}{\partial t}(\overline{\rho' u_i'}) \tag{15}$$

and the continuity equation (9) multiplied by the appropriate velocity component $\bar{u}$ or $\bar{v}$. The resulting system of equations governing the flow are:

$$\frac{\partial\bar{\rho}}{\partial t} + \frac{\partial}{\partial x}(\overline{\rho u}) + \frac{\partial}{\partial y}(\overline{\rho v}) = 0 \tag{16}$$

$$\frac{\partial}{\partial t}(\overline{\rho u}) + \frac{\partial}{\partial x}(\overline{\rho u}\,\bar{u}) + \frac{\partial}{\partial y}(\overline{\rho v}\,\bar{u}) + \frac{\partial\bar{p}}{\partial x} = \frac{\partial}{\partial x}\bar{\tau}_{xx}^{\ t} + \frac{\partial}{\partial y}\bar{\tau}_{yx}^{\ t} \tag{17}$$

$$\frac{\partial}{\partial t}(\overline{\rho v}) + \frac{\partial}{\partial x}(\overline{\rho u}\,\bar{v}) + \frac{\partial}{\partial y}(\overline{\rho v}\,\bar{v}) + \frac{\partial\bar{p}}{\partial y} = \frac{\partial}{\partial y}\bar{\tau}_{yy}^{\ t} + \frac{\partial}{\partial x}\bar{\tau}_{xy}^{\ t} \tag{18}$$

$$\bar{p} = \bar{\rho}RT \tag{19}$$

These equations, as expressed, are in their least objectional form from the standpoint of the turbulent nature of the flow, in that the "mean" of products of the fluctuating quantities do not appear explicitly. Equations (17) and (18) are simply the Navier-Stokes equations for the "mean" flow written in the form of equations (6) and (7) together with the definitions, equations (14). The flow has been restricted to two dimensions, turbulence has been described in terms of fluctuations about "mean" values, and heat transfer has been neglected in formulating the problem. Even so, the set of equations are not amendable to exact solution and recourse must be made to approximate methods.

INTEGRAL EQUATIONS

The present problem requires information pertaining to the effect of a turbulent boundary layer on the pressure distribution over a sinusoidally oscillating wall. This information cannot be obtained from exact solution of the governing equations. It remains then to obtain an approximation to this effect by considering the boundary layer in an approximate manner. Such an approach is formulated in the literature for thin boundary layers and is referred to as the integral equation approach. The integral equation resulting from the thin boundary layer momentum equation, in essence, allows the boundary layer to satisfy the governing equations on the average, but not in detail throughout the boundary layer. One proceeds with this method by specifying the potential flow or a suitable approximation, knowledge of which allows calculation of the boundary layer. The boundary layer calculation provides information relating to skin friction and defined boundary layer properties such as the various boundary layer thicknesses. In principle, these results may be used to recalculate the potential flow with successive approximations to the interaction of the boundary layer and potential flow.

Inasmuch as the thin boundary layer approach is an approximate formulation with inherent approximations that explicitly ignore the pressure variations of interest, namely the variations between pressures on the surface and those in the local potential flow, it would seem desirable to formulate and examine the viscous wavy wall problem in terms of "exact" integral equations. This may be accomplished utilizing equations (17) and (18) without assuming a thin boundary layer but retaining the basic approximate concept of the presence of a boundary layer and the boundary condition at its outer edge.

Accordingly, one states that at $y = \delta$,

$$\bar{u} = U$$

$$\frac{\partial \bar{u}}{\partial y} = 0$$

$$\frac{\partial \bar{u}}{\partial x} = U'$$

$$\frac{\partial \bar{p}}{\partial x} = \left.\frac{\partial \bar{p}}{\partial x}\right|_{y=\delta} = -\rho UU' - \rho \frac{\partial U}{\partial t} \tag{20}$$

where δ is interpreted as the boundary layer thickness and the prime denotes differentiation with respect to x.

To express equations (17) and (18) in an integral formulation, equation (18) is differentiated with respect to x and integrated over y from the limit $y = \delta$ to $y = y$ utilizing the potential flow conditions at $y = \delta$. The resultant expression for $\partial\bar{p}/\partial x$ may then be used in equation (17) to eliminate the pressure gradient. Then equation (17) may be integrated, at least theoretically, across the entire boundary layer to provide the "exact" boundary layer integral equation analogous to the conventional thin boundary layer integral equation. In addition, the result of the treatment of equation (18) to obtain $\partial\bar{p}/\partial x$ describes the variation of the pressure gradient across the boundary layer.

Following this approach, differentiation of equation (18) with respect to x yields

$$\frac{\partial^2\bar{p}}{\partial x\partial y} = \frac{\partial^2}{\partial x\partial y}\bar{\tau}_{yy}^{\ t} + \frac{\partial^2}{\partial x^2}\bar{\tau}_{xy}^{\ t} - \frac{\partial^2\overline{\rho v}}{\partial x\partial t} - \frac{\partial^2}{\partial x^2}(\overline{\rho u}\ \bar{v}) - \frac{\partial^2}{\partial x\partial y}(\overline{\rho v}\ \bar{v}) \tag{21}$$

where

$$\frac{\partial^2\bar{p}}{\partial x\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial\bar{p}}{\partial x}\right)$$

Integration of equation (21) over the interval δ to y, together with equations (20) and

$$\int_\delta^y \frac{\partial}{\partial y}\left(\frac{\partial\bar{p}}{\partial x}\right) dy = \left.\frac{\partial\bar{p}}{\partial x}\right|_y + \rho UU' + \rho\frac{\partial U}{\partial t} \tag{22}$$

$$\int_\delta^y \frac{\partial^2\bar{\tau}_{yy}^{\ t}}{\partial x\partial y}\, dy = \left.\frac{\partial\bar{\tau}_{yy}^{\ t}}{\partial x}\right|_y - \left.\frac{\partial\bar{\tau}_{yy}^{\ t}}{\partial x}\right|_\delta \tag{23}$$

with

$$\left.\frac{\partial\bar{\tau}_{yy}^{\ t}}{\partial x}\right|_{y=\delta} = 0$$

yields

$$\frac{\partial\bar{p}}{\partial x} = -\rho\frac{\partial U}{\partial t} - \rho UU' + \frac{\partial}{\partial x}\bar{\tau}_{yy}^{\ t} - \int_\delta^y \frac{\partial^2\overline{\rho v}}{\partial x\partial t}\, dy + \int_\delta^y \frac{\partial^2}{\partial x^2}\bar{\tau}_{xy}^{\ t}\, dy$$
$$- \int_\delta^y \frac{\partial^2}{\partial x^2}(\overline{\rho u}\ \bar{v})\, dy - \int_\delta^y \frac{\partial^2}{\partial x\partial y}(\overline{\rho v}\ \bar{v})\, dy \tag{24}$$

For convenience, the velocities in the various terms of equations (17) and (24) are non-dimensionalized by division by the value of ρU^2 at the edge of the boundary layer. The resulting governing equations before integration over the boundary layer may be rewritten as follows after considerable manipulation and some rearrangement as

$$\frac{1}{U}\left\{\frac{\partial}{\partial t}\left(\frac{\overline{\rho u}}{\rho U}\right) + \frac{\overline{\rho u}}{\rho U}\left[\frac{1}{\rho U}\frac{\partial(\rho U)}{\partial t}\right]\right\} + \frac{\overline{\rho u}}{\rho U}\frac{\bar{u}}{U}\left[\frac{(\rho U^2)'}{\rho U^2}\right] + \frac{\partial}{\partial x}\left(\frac{\overline{\rho u}}{\rho U}\frac{\bar{u}}{U}\right)$$

$$+ \frac{\partial}{\partial y}\left(\frac{\overline{\rho v}}{\rho U}\frac{\bar{u}}{U}\right) + \frac{\partial\bar{p}/\partial x}{\rho U^2} = \frac{(\rho U^2)'}{\rho U^2}\frac{\bar{\tau}_{xx}^{\,t}}{\rho U^2}$$

$$+ \frac{\partial}{\partial x}\frac{\bar{\tau}_{xx}^{\,t}}{\rho U^2} + \frac{\partial}{\partial y}\frac{\bar{\tau}_{yx}^{\,t}}{\rho U^2} \tag{25}$$

and

$$\frac{\partial\bar{p}/\partial x}{\rho U^2} = -\frac{1}{U^2}\frac{\partial U}{\partial t} - \int_\delta^y \left\{\frac{(\rho U^2)'}{\rho U^2}\frac{1}{U}\left[\frac{\partial}{\partial t}\left(\frac{\overline{\rho v}}{\rho U}\right) + \frac{\overline{\rho v}}{\rho U}\left(\frac{1}{\rho U}\frac{\partial(\rho U)}{\partial t}\right)\right]\right.$$

$$\left. + \frac{\partial}{\partial x}\left\{\frac{1}{U}\left[\frac{\partial}{\partial t}\left(\frac{\overline{\rho v}}{\rho U}\right) + \frac{\overline{\rho v}}{\rho U}\left(\frac{1}{\rho U}\frac{\partial(\rho U)}{\partial t}\right)\right]\right\}\right\} dy - \frac{U'}{U} + \frac{(\rho U^2)'}{\rho U^2}\frac{\bar{\tau}_{yy}^{\,t}}{\rho U^2}$$

$$+ \frac{\partial}{\partial x}\frac{\bar{\tau}_{yy}^{\,t}}{\rho U^2} + \int_\delta^y \left[\frac{(\rho U^2)''}{\rho U^2}\frac{\bar{\tau}_{xy}^{\,t}}{\rho U^2} + 2\,\frac{(\rho U^2)'}{\rho U^2}\frac{\partial}{\partial x}\frac{\bar{\tau}_{xy}^{\,t}}{\rho U^2} + \frac{\partial^2}{\partial x^2}\frac{\bar{\tau}_{xy}^{\,t}}{\rho U^2}\right] dy$$

$$- \int_\delta^y \left[\frac{(\rho U^2)''}{\rho U^2}\frac{\overline{\rho u}}{\rho U}\frac{\bar{v}}{U} + 2\,\frac{(\rho U^2)'}{\rho U^2}\frac{\partial}{\partial x}\left(\frac{\overline{\rho u}}{\rho U}\frac{\bar{v}}{U}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\overline{\rho u}}{\rho U}\frac{\bar{v}}{U}\right)\right] dy$$

$$- \int_\delta^y \left\{\frac{(\rho U^2)'}{\rho U^2}\left[\frac{\partial}{\partial y}\left(\frac{\overline{\rho v}}{\rho U}\frac{\bar{v}}{U}\right)\right] + \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\overline{\rho v}}{\rho U}\frac{\bar{v}}{U}\right)\right]\right\} dy \tag{26}$$

Equation (25), after substitution of the pressure gradient from equation (26) and integration over the boundary layer, replaces the conventional thin boundary layer integral momentum equation which defines the relation existing between boundary layer displacement thickness, momentum thickness, skin friction and pressure gradient. Whereas in the thin boundary layer approach the pressure gradient must be prescribed by the potential flow, the present formulation yields an integral representation of the change in the pressure gradient across the boundary layer from equation (26) if the limits of the integrals in this equation are taken from the surface to the edge of the boundary layer.

For the sake of brevity, the integral form of equation (25) is not given at this point; no simplification is effected by indication of the integration and knowledge of the pressure gradient and skin friction would be required in order to utilize this particular equation to any extent.

The integral equation approach to this problem may be briefly stated, in view of the present formulation, as follows. Equation (26) provides a statement of the pressure gradient in the form of an integral equation requiring knowledge of the flow variables throughout the boundary layer to effect integration and knowledge of the potential flow to complete the computation of the pressure gradient at the surface. The integral form of equation (25), on the other hand, should provide information as to the dependence of the potential flow on the boundary layer thickness, this computation again requiring knowledge of the boundary layer flow variables, skin friction, and the variation of the pressure gradient through the boundary layer as described by equation (26). It is anticipated then that the boundary layer flow variables may be described in a number of different ways or by means of different models. In that the "exact" model is not known, the value of the results is dependent upon the model chosen.

DISCUSSION

Any discussion of the integral equation approach, which at its present stage results in equations (25) and (26), must make note of the obvious fact that this approach does not provide direct simplification to the problem but simply provides a more convenient method to account for the presence of the boundary layer. No detailed study of the flow in the boundary layer is possible by this method; instead, details of the boundary layer flow or their approximations are required in order to utilize the approach. In this connection, it appears that the integral method of representation of the boundary layer possesses some advantage over other approximate methods in that it states explicitly what approximations or representations of the boundary layer flow are required and it shows how these approximations enter into computations of the boundary layer effects on pressure and the representation of the potential flow.

On the other hand, the integral formulation, equations (25) and (26), is complicated by the large number of approximations of the boundary layer flow that are required. Obviously, a reduction in the number of such approximations or terms in the integral equations is desired. This consideration is currently being studied as well as the evaluation of the various boundary layer models that have been proposed in the literature, particularly those due to McClure (3), (4), (5), Miles (6), Mercer (7), Benjamin (8), and Fung (9).

One obvious simplification to the general formulation of the integral approach is the restriction of the regular perturbations due to the wall oscillation to small amplitudes. In effect, this assumption allows one to define all of the unsteady "mean" flow variables as the sum of a steady flow contribution due to the flow past a stationary flat plate plus a regular perturbation due to the wall oscillation. Further, it would seem reasonable to assume that for panels that are short relative to the main flow direction, variations of the steady flow contribution in this direction should be negligible, particularly for thick boundary layers. A study of these simplifications is in progress and it is anticipated that further details will be available in the second quarterly report.

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PREFACE

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The principal investigator of the program is Dr. E. F. E. Zeydel.

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u,v = x and y velocity components, respectively

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τ_{ij} = stress in j^{th} direction on i^{th} plane

ρ = fluid density

μ = coefficient of viscosity

Superscripts and other:

$(\ )'$ = fluctuation component or differentiation with respect to x

$(\bar{\ })$ = "mean" or steady "mean" component

$(\tilde{\tilde{\ }})$ = regular fluctuation (unsteady "mean" component)

I. INTRODUCTION

In this project period a theoretical analysis was developed to estimate the pressure distribution on a stationary wavy wall of infinite extent in the streamwise direction with a thick boundary layer. The analysis is based on the assumption that the boundary layer can be represented by a two-dimensional mean laminar flow with given velocity profile $U(y)$ as is done in [1]*.

In addition, the mean flow equations derived in the previous progress report (11 June 1965 to 10 September 1965) have been extended to incorporate the effects of small perturbations on a mean flow condition. The resulting equations have been restricted to flows in which the mean steady component is independent of the streamwise coordinate.

II. THE PRESSURE DISTRIBUTION ON A STATIONARY WAVY WALL WITH A THICK BOUNDARY LAYER

Consider a two-dimensional mean laminar flow in the x-direction with a given mean velocity profile $U(y)$. We assume that the mean flow is a boundary layer flow, which joins the uniform external flow, U_m, at $y = \delta$ (see Fig. 1).

Let the stationary wavy wall boundary at $y = 0$ be given by

$$w_o = A_o e^{i\lambda x}$$

The problem thus becomes to determine the pressure perturbation, p_o, at the plane $y = 0$ and within the boundary layer in terms of w_o.

For simplicity, we divide the boundary layer δ into m sublayers of thicknesses $\delta_o, \delta_1, \ldots, \delta_i, \ldots, \delta_{m-1}$ and assume that in each sublayer the mean velocity and density is uniform. We assume also that the amplitude A_o is small compared to the thickness of each sublayer and that the perturbed flows in the sublayers and in the external flow are isentropic and irrotational. The linearized equations for the perturbation velocity potential φ can then be used to obtain the perturbation velocity and pressure fields within each sublayer. Since the external flow velocity, U_m, is supersonic, the mean flow velocity in the sublayers are either sub- or supersonic.

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*Numbers in brackets refer to the bibliography.

The expressions between the streamlines and pressures at the boundaries of a sub- or supersonic sublayer can be obtained as follows.

Let the uniform mean flow velocity, the fluid density, and Mach number of the i^{th} sublayer of thickness, δ_i , be given by U_i , ρ_i and M_i , respectively (see Fig. 1).

The linearized equation for the velocity potential, φ_i , in the sublayer is

$$\left(1 - M_i^2\right) \varphi_{ixx} + \varphi_{iy_iy_i} = 0 \tag{1}$$

The boundary conditions are $(-\infty < x < \infty)$.

$$\left.\frac{\partial\varphi_i}{\partial y_i}\right|_{y_i=0} = U_i \frac{\partial w_i}{\partial x_i} \tag{2}$$

$$\left.\frac{\partial\varphi_i}{\partial y_i}\right|_{y_i=\delta_i} = U_i \frac{\partial w_{i+1}}{\partial x_i} \tag{3}$$

$$-\rho_i U_i \left.\frac{\partial\varphi_i}{\partial x}\right|_{y_i=0} = p_i \tag{4}$$

$$-\rho_i U_i \left.\frac{\partial\varphi_i}{\partial x}\right|_{y_i=\delta_i} = p_{i+1} \tag{5}$$

where

w_i = streamline at $y_i = 0$

w_{i+1} = streamline at $y_i = \delta_i$

p_i = pressure perturbation at $y_i = 0$

p_{i+1} = pressure perturbation at $y_i = \delta_i$

Let

$$w_i = A_i \, e^{i\lambda x} \tag{6}$$

$$w_{i+1} = A_{i+1} \, e^{i\lambda x} \tag{7}$$

$$p_i = -\rho_i U_i^{\,2} \, P_i \, e^{i\lambda x} \tag{8}$$

$$p_{i+1} = -\rho_{i+1} \, U_{i+1}^{\,2} \, P_{i+1} \, e^{i\lambda x} \tag{9}$$

where A_i , A_{i+1} , P_i , and P_{i+1} are in general complex quantities.

If U_i is subsonic, so that $M_i < 1$, equation (1) has the solution

$$\varphi_i = U_i\left(C_i \, e^{-\lambda\gamma_i y_i} + D_i \, e^{+\lambda\gamma_i y_i}\right) e^{i\lambda x} \tag{10}$$

where

$$\gamma_i = \sqrt{1 - M_i^{\,2}} \tag{11}$$

Using (2) - (4), (6) - (8), and (10), we eliminate C_i and D_i and find

$$A_{i+1} = \frac{e^{-\alpha_i} + e^{+\alpha_i}}{2} \, A_i + \frac{\gamma_i}{\lambda} \, \frac{e^{-\alpha_i} - e^{+\alpha_i}}{2} \, P_i \tag{12}$$

where

$$\alpha_i = \lambda\gamma_i\delta_i \tag{13}$$

Similarly, using (2), (4) - (6) and (8) - (10), yields

$$P_{i+1} = \frac{\rho_i U_i^{\,2}}{\rho_{i+1} U_{i+1}^{\,2}} \, \frac{\lambda}{\gamma_i} \, \frac{e^{-\alpha_i} - e^{+\alpha_i}}{2} \, A_i + \frac{\rho_i U_i^{\,2}}{\rho_{i+1} U_{i+1}^{\,2}} \, \frac{e^{-\alpha_i} + e^{-\alpha_i}}{2} \, P_i \tag{14}$$

Equation (12) and (14) give the desired relation between the streamline and pressure coefficients at the boundaries of a subsonic sublayer.

If U_i is supersonic, so that $M_i > 1$, equation (1) has the solution

$$\varphi_i = U_i\left(E_i\, e^{-i\lambda\beta_i y_i} + F_i\, e^{+i\lambda\beta_i y_i}\right) e^{i\lambda x} \tag{15}$$

where

$$\beta_i = \sqrt{M_i^{\,2} - 1} \tag{16}$$

Again we eliminate E_i and F_i , using (2) - (4), (6) - (8) and (15) and find

$$A_{i+1} = \frac{e^{-\bar{\alpha}_i} + e^{+\bar{\alpha}_i}}{2} A_i + \frac{i\beta_i}{\lambda} \frac{e^{-\bar{\alpha}_i} - e^{+\bar{\alpha}_i}}{2} P_i \tag{17}$$

where

$$\bar{\alpha}_i = i\lambda\beta_i\delta_i \tag{18}$$

Similarly, using (2), (4) - (6), (8), (9), and (15), gives

$$P_{i+1} = - \frac{\rho_i U_i^{\,2}}{\rho_{i+1}U_{i+1}^{\,2}} \frac{i\lambda}{\beta_i} \frac{e^{-\bar{\alpha}_i} - e^{+\bar{\alpha}_i}}{2} A_i + \frac{\rho_i U_i^{\,2}}{\rho_{i+1}U_{i+1}^{\,2}} \frac{e^{-\bar{\alpha}_i} - e^{+\bar{\alpha}_i}}{2} P_i \tag{19}$$

Note that (17) and (19) can be used to calculate A_{i+1} and P_{i+1} if the sublayer is supersonic and A_i and P_i are given.

If U_i equals the speed of sound so that $M_i = 1$, we can use either (12) and (14) or (17) and (19) to calculate A_{i+1} and P_{i+1} . It is interesting to note that the coefficients of A_i and P_i in these equations remain finite at $M_i = 1$, since

$$\lim_{\gamma_i \to 0} \frac{e^{-\alpha_i} - e^{+\alpha_i}}{\gamma_i} \quad \text{and} \quad \lim_{\beta_i \to 0} \frac{i\left(e^{-\bar{\alpha}_i} - e^{+\bar{\alpha}_i}\right)}{\beta_i}$$

are finite.

Next, the expressions derived above are used to calculate the streamline and perturbation pressure coefficients at the edge of the boundary layer in terms of those at the surface.

For clarity, we write (12), (14) and (17), (19) in matrix form.

$$\begin{vmatrix} A_{i+1} \\ P_{i+1} \end{vmatrix} = \left[G_i\right] \begin{vmatrix} A_i \\ P_i \end{vmatrix} \tag{20}$$

where the elements of G_i are given by

$$g_{i_{11}} = \frac{e^{-\alpha_i} + e^{+\alpha_i}}{2}$$

$$g_{i_{12}} = \frac{\gamma_i}{\lambda} \frac{e^{-\alpha_i} - e^{+\alpha_i}}{2}$$

$$g_{i_{21}} = \frac{\rho_i U_i^{\,2}}{\rho_{i+1} U_{i+1}^{\,2}} \frac{\lambda}{\gamma_i} \frac{e^{-\alpha_i} - e^{+\alpha_i}}{2}$$

$$g_{i_{22}} = \frac{\rho_i U_i^{\,2}}{\rho_{i+1} U_{i+1}^{\,2}} \frac{e^{-\alpha_i} + e^{+\alpha_i}}{2} \quad \text{when} \quad M_i \leq 1 \tag{21}$$

and

$$g_{i_{11}} = \cos \lambda\, \beta_i \delta_i$$

$$g_{i_{12}} = + \frac{\beta_i}{\lambda} \sin \lambda\, \beta_i \delta_i$$

$$g_{i_{21}} = - \frac{\rho_i U_i^{\,2}}{\rho_{i+1} U_{i+1}^{\,2}} \frac{\lambda}{\beta_i} \sin \lambda\, \beta_i \delta_i$$

$$g_{i_{22}} = \frac{\rho_i U_i^2}{\rho_{i+1} U_{i+1}^2} \cos \lambda \, \beta_i \delta_i \quad \text{when} \quad M_i > 1 \tag{22}$$

Then

$$\begin{vmatrix} A_m \\ P_m \end{vmatrix} = \begin{bmatrix} H \end{bmatrix} \begin{vmatrix} A_o \\ P_o \end{vmatrix} \tag{23}$$

where

$$\begin{bmatrix} H \end{bmatrix} = \begin{bmatrix} G_{m-1} \end{bmatrix} \begin{bmatrix} G_{m-2} \end{bmatrix} \cdot \cdot \cdot \cdot \begin{bmatrix} G_0 \end{bmatrix} \tag{24}$$

Since the flow external to the boundary layer is supersonic and uniform (U_m = constant , $M_m > 1$) , the velocity potential is given by

$$\varphi_m = U_m E_m e^{-i\lambda\beta_m y_m} e^{i\lambda x} \tag{25}$$

where

$$\beta_m = \sqrt{M_m^2 - 1} \tag{26}$$

Using (2) and (6), we find

$$E_m = - \frac{A_m}{\beta_m} \tag{27}$$

so that from (4) and (8) there follows

$$P_m = -i \frac{\lambda}{\beta_m} A_m \tag{28}$$

The pressure coefficient, P_o , at the surface of the wavy wall is obtained from (23) and (24) using (27) and (28).

$$P_o = - \frac{h_{21} + i(\lambda/\beta_m)h_{11}}{h_{22} + i(\lambda/\beta_m)h_{12}} A_o \tag{29}$$

where h_{11} , h_{12} , h_{21} , and h_{22} are the elements of the matrix H.

The perturbation pressure at the surface of the stationary wavy wall is given by

$$p_o = -\rho_o U_o^2 P_o \, e^{i\lambda x} \tag{30}$$

An analysis is presently underway to examine the perturbation pressure on stationary wavy walls of infinite extent in the x-direction with various wave lengths, using the boundary layer profiles obtained from the Ames Research Center.

An attempt is also made to extend the analysis to the unsteady case and the finite length case.

III. MEAN FLOW EQUATIONS WITH SMALL PERTURBATIONS

The first quarterly progress report (11 June 1965 to 10 September 1965) concluded with the presentation of an integral formulation to the general problem of the unsteady turbulent flow over an oscillating wall. In their general form these equations do not explicitly yield any information as to the influence of the wall oscillations on the pressure distribution over the wall; further, the equations are complicated by the retention of all terms from the corresponding Navier-Stokes equations for the "mean" flow.

An obvious simplification to the general formulation may be accomplished by incorporating two assumptions:

(a) the disturbances due to harmonic wall oscillations are assumed to be small; in effect then, the flow is described as the sum of a "mean" steady flow plus a "mean" regular fluctuation which permits all flow variables to be defined as follows:

$$\bar{(\;)}_{\text{"mean" flow}} = \bar{(\;)}_{\text{steady "mean" flow}} + \tilde{\tilde{(\;)}}_{\substack{\text{unsteady regular}\\ \text{fluctuation}}}$$

(b) the variations of the steady "mean" flow components (that is, the flow which would be present if wall oscillations did not exist) in the panel chordwise direction are neglected.

These assumptions are consistent with the approach used in inviscid flows over wavy walls of infinite extent wherein mathematical simplification is desirable and permissible. Further simplifications in the inviscid theories, by an order of magnitude analysis, leads to the linearized theories for subsonic and supersonic flow and the nonlinear correction theory for transonic flow [2].

A detailed discussion of these assumptions and their consequences is given in the following development. It is convenient to pursue these simplifications from the standpoint of the original form of the governing equations, equations (16), (17), and (18) of the first quarterly progress report.

Governing Equations

The Navier-Stokes equations for the "mean" flow in two dimensions together with the appropriate continuity equation are:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x} (\overline{\rho u}) + \frac{\partial}{\partial y} (\overline{\rho v}) = 0 \tag{31}$$

$$\frac{\partial}{\partial t} (\overline{\rho u}) + \frac{\partial}{\partial x} (\overline{\rho u}\ \bar{u}) + \frac{\partial}{\partial y} (\overline{\rho v}\ \bar{u}) + \frac{\partial \bar{p}}{\partial x}$$

$$= \frac{\partial}{\partial x} \bar{\tau}_{xx}{}^{t} + \frac{\partial}{\partial y} \bar{\tau}_{yx}{}^{t} \tag{32}$$

$$\frac{\partial}{\partial t} (\overline{\rho v}) + \frac{\partial}{\partial x} (\overline{\rho u}\ \bar{v}) + \frac{\partial}{\partial y} (\overline{\rho v}\ \bar{v}) + \frac{\partial \bar{p}}{\partial y}$$

$$= \frac{\partial}{\partial y} \bar{\tau}_{yy}{}^{t} + \frac{\partial}{\partial x} \bar{\tau}_{xy}{}^{t} \tag{33}$$

Steady Mean Turbulent Flow

The neglect of chordwise variations of flow variables in the "mean" steady flow component requires that

$$\frac{\partial (\ )}{\partial x} = 0 \tag{34}$$

in addition, $\frac{\partial (\ )}{\partial t} = 0$ by definition and the governing equations (31) - (33) then reduce to the simple equations that follow:

continuity $$\frac{\partial}{\partial y} (\overline{\rho v}) = 0 \tag{35}$$

x-momentum $$\frac{\partial}{\partial y} (\overline{\rho v}\ \bar{u}) = \frac{\partial}{\partial y} \bar{\tau}_{yx}{}^{t} \tag{36}$$

y-momentum $$\frac{\partial}{\partial y}\left(\overline{\rho v}\ \bar{v}\right) + \frac{\partial \bar{p}}{\partial y} = \frac{\partial}{\partial y}\bar{\tau}_{yy}^{\ t} \tag{37}$$

A detailed examination of these equations yields some insight into the structure of the steady "mean" flow component, which is desirable, inasmuch as deviations from this state must be due to the wall oscillation. Consideration of each of these equations follows:

<u>Continuity, Steady Mean Flow:</u>

As a result of equation (35) and the boundary condition

$$\overline{\rho v}\Big|_{wall} = 0$$

one obtains

$$\overline{\rho v} = 0 \tag{38}$$

Further, by definition

$$\overline{\rho v} = \bar{\rho}\bar{v} + \overline{\rho' v'}$$

so that

$$\bar{\rho}\bar{v} = -\ \overline{\rho' v'} \tag{39}$$

<u>x-momentum, Steady Mean Flow:</u>

By definition

$$\bar{\tau}_{yx}^{\ t} \equiv \bar{\tau}_{yx} - \overline{(\rho v)' u'}$$

$$\equiv \mu\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) - \overline{(\rho v)' u'}$$

so by virtue of equation (35) or (38) and the boundary condition

$$\overline{(\rho v)' u'}\Big|_{wall} = 0$$

equation (36) yields for the assumed steady "mean" flow:

$$\mu \frac{\partial \bar{u}}{\partial y} = \mu \frac{\partial \bar{u}}{\partial y}\Big|_{wall} + \overline{(\rho v)' u'} \tag{40}$$

y-momentum, Steady Mean Flow:

By definition

$$\bar{\tau}_{yy}{}^{t} = \bar{\tau}_{yy} - \overline{(\rho v)'v'}$$

$$= 2\mu \frac{\partial \bar{v}}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) - \overline{(\rho v)'v'}$$

so by virtue of equation (35) or (38) and the boundary condition

$$\overline{(\rho v)'v'}\Big|_{wall} = 0$$

equation (37) yields for the assumed steady "mean" flow:

$$\bar{p} = \bar{p}_{wall} + \frac{4}{3}\mu\left(\frac{\partial \bar{v}}{\partial y} - \frac{\partial \bar{v}}{\partial y}\Big|_{wall}\right) - \overline{(\rho v)'v'} \qquad (41)$$

The resulting steady "mean" flow is then the compressible counterpart of a steady, parallel, incompressible shear flow. While a solution for the "mean" steady flow is not sought, it is both interesting and necessary to formulate the consequences of the neglect of chordwise variations of flow as given by equations (38), (39), 40), and (41). In particular, it is evident that a pressure variation through the thick turbulent boundary layer is present and contains, according to equation (41), contributions due the turbulent character of the flow and a combination viscous compressibility term. It is intuitively assumed that this pressure variation is small; in any case, it is the deviation from this state, due to the wall oscillations, in the presence of the boundary layer that is sought.

Small Perturbation Equations

The results of assuming small perturbations to the steady "mean" turbulent flow due to regular wall oscillations are found by rewriting the governing equations (31) - (33) replacing the total "mean" flow variables with their corresponding sum of steady and regular fluctuations. The deviations from the steady "mean" flow may then be examined in the form of governing equations for the perturbations inasmuch as the steady "mean" flow satisfies equations (35) - (37).

Continuity, Perturbed Mean Flow

Inasmuch as no products of variables appear in the continuity equation (31), the corresponding perturbation equation is given by

$$\frac{\partial \tilde{\tilde{\rho}}}{\partial t} + \frac{\partial}{\partial x} \left(\widetilde{\widetilde{\rho u}}\right) + \frac{\partial}{\partial y} \left(\widetilde{\widetilde{\rho v}}\right) = 0 \tag{42}$$

x-momentum, Perturbed Mean Flow

Equation (32) becomes upon substitution

$$\frac{\partial}{\partial t} \left(\overline{\rho u}\right) + \frac{\partial}{\partial t} \left(\widetilde{\widetilde{\rho u}}\right) + \frac{\partial}{\partial x} \left[\left(\overline{\rho u} + \widetilde{\widetilde{\rho u}}\right)\left(\bar{u} + \tilde{\tilde{u}}\right)\right] + \frac{\partial}{\partial y} \left[\left(\overline{\rho v} + \widetilde{\widetilde{\rho v}}\right)\left(\bar{u} + \tilde{\tilde{u}}\right)\right]$$

$$+ \frac{\partial \bar{p}}{\partial x} + \frac{\partial \tilde{\tilde{p}}}{\partial x} = \frac{\partial}{\partial x} \left(\bar{\tau}_{xx}{}^{t} + \tilde{\tilde{\tau}}_{xx}{}^{t}\right) + \frac{\partial}{\partial y} \left(\bar{\tau}_{yx}{}^{t} + \tilde{\tilde{\tau}}_{yx}{}^{t}\right)$$

where the symbol ($\bar{\ }$) now refers only to the steady component of the "mean" flow. Neglecting products of small terms and utilizing equations (35) and (36), this equation reduces to the corresponding x-momentum perturbation equation:

$$\frac{\partial}{\partial t} \left(\widetilde{\widetilde{\rho u}}\right) + \overline{\rho u}\, \frac{\partial \tilde{\tilde{u}}}{\partial x} + \bar{u}\, \frac{\partial \widetilde{\widetilde{\rho u}}}{\partial x} + \widetilde{\widetilde{\rho v}}\, \frac{\partial \bar{u}}{\partial y} + \bar{u}\, \frac{\partial \widetilde{\widetilde{\rho v}}}{\partial y} + \frac{\partial \tilde{\tilde{p}}}{\partial x} = \frac{\partial}{\partial x} \tilde{\tilde{\tau}}_{xx}{}^{t} + \frac{\partial}{\partial y} \tilde{\tilde{\tau}}_{yx}{}^{t} \tag{43}$$

y-Momentum, Perturbed Mean Flow

Similarly equation (33) is expanded by substitution to obtain:

$$\frac{\partial}{\partial t} \left(\overline{\rho v}\right) + \frac{\partial}{\partial t} \left(\widetilde{\widetilde{\rho v}}\right) + \frac{\partial}{\partial x} \left[\left(\overline{\rho u} + \widetilde{\widetilde{\rho u}}\right)\left(\bar{v} + \tilde{\tilde{v}}\right)\right] + \frac{\partial}{\partial y} \left[\left(\overline{\rho v} + \widetilde{\widetilde{\rho v}}\right)\left(\bar{v} + \tilde{\tilde{v}}\right)\right]$$

$$+ \frac{\partial \bar{p}}{\partial y} + \frac{\partial \tilde{\tilde{p}}}{\partial y} = \frac{\partial}{\partial y} \left(\bar{\tau}_{yy}{}^{t} + \tilde{\tilde{\tau}}_{yy}{}^{t}\right) + \frac{\partial}{\partial x} \left(\bar{\tau}_{xy}{}^{t} + \tilde{\tilde{\tau}}_{xy}{}^{t}\right)$$

and reduced in like manner to the y-momentum equation:

$$\frac{\partial}{\partial t}\left(\widetilde{\widetilde{\rho v}}\right) + \overline{\rho u}\,\frac{\partial \tilde{\tilde{v}}}{\partial x} + \bar{v}\,\frac{\partial \widetilde{\widetilde{\rho u}}}{\partial x} + \widetilde{\widetilde{\rho v}}\,\frac{\partial \bar{v}}{\partial y} + \bar{v}\,\frac{\partial \widetilde{\widetilde{\rho v}}}{\partial y} + \frac{\partial \tilde{\tilde{p}}}{\partial y} = \frac{\partial}{\partial y}\,\tilde{\tilde{\tau}}_{yy}{}^{t} + \frac{\partial}{\partial x}\,\tilde{\tilde{\tau}}_{xy}{}^{t} \tag{44}$$

Perturbation Equations, Alternate Forms

Collecting the perturbation equations, one has

$$\frac{\partial \tilde{\tilde{\rho}}}{\partial t} + \frac{\partial}{\partial x}\left(\widetilde{\widetilde{\rho u}}\right) + \frac{\partial}{\partial y}\left(\widetilde{\widetilde{\rho v}}\right) = 0 \tag{42}$$

$$\frac{\partial}{\partial t}\left(\widetilde{\widetilde{\rho u}}\right) + \overline{\rho u}\,\frac{\partial \tilde{\tilde{u}}}{\partial x} + \bar{u}\,\frac{\partial \widetilde{\widetilde{\rho u}}}{\partial x} + \widetilde{\widetilde{\rho v}}\,\frac{\partial \bar{u}}{\partial y} + \bar{u}\,\frac{\partial \widetilde{\widetilde{\rho v}}}{\partial y} + \frac{\partial \tilde{\tilde{p}}}{\partial x}$$

$$= \frac{\partial}{\partial x}\,\tilde{\tilde{\tau}}_{xx}{}^{t} + \frac{\partial}{\partial y}\,\tilde{\tilde{\tau}}_{yx}{}^{t} \tag{43}$$

$$\frac{\partial}{\partial t}\left(\widetilde{\widetilde{\rho v}}\right) + \overline{\rho u}\,\frac{\partial \tilde{\tilde{v}}}{\partial x} + \bar{v}\,\frac{\partial \widetilde{\widetilde{\rho u}}}{\partial x} + \widetilde{\widetilde{\rho u}}\,\frac{\partial \bar{v}}{\partial y} + \bar{v}\,\frac{\partial \widetilde{\widetilde{\rho v}}}{\partial y} + \frac{\partial \tilde{\tilde{p}}}{\partial y}$$

$$= \frac{\partial}{\partial y}\,\tilde{\tilde{\tau}}_{yy}{}^{t} + \frac{\partial}{\partial x}\,\tilde{\tilde{\tau}}_{xy}{}^{t} \tag{44}$$

Alternately, using equation (42), equations (43) and (44) may be re-written as

$$\frac{\partial}{\partial t}\left(\widetilde{\widetilde{\rho u}}\right) - \bar{u}\,\frac{\partial \tilde{\tilde{\rho}}}{\partial t} + \overline{\rho u}\,\frac{\partial \tilde{\tilde{u}}}{\partial x} + \widetilde{\widetilde{\rho v}}\,\frac{\partial \bar{u}}{\partial y} + \frac{\partial \tilde{\tilde{p}}}{\partial x}$$

$$= \frac{\partial}{\partial x}\,\tilde{\tilde{\tau}}_{xx}{}^{t} + \frac{\partial}{\partial y}\,\tilde{\tilde{\tau}}_{yx}{}^{t} \tag{43a}$$

$$\frac{\partial}{\partial t}\left(\widetilde{\widetilde{\rho v}}\right) - \bar{v}\,\frac{\partial \tilde{\tilde{\rho}}}{\partial t} + \overline{\rho u}\,\frac{\partial \tilde{\tilde{v}}}{\partial x} + \widetilde{\widetilde{\rho v}}\,\frac{\partial \bar{v}}{\partial y} + \frac{\partial \tilde{\tilde{p}}}{\partial y}$$

$$= \frac{\partial}{\partial y}\,\tilde{\tilde{\tau}}_{yy}{}^{t} + \frac{\partial}{\partial x}\,\tilde{\tilde{\tau}}_{xy}{}^{t} \tag{44a}$$

Expanding the unsteady terms in equations (43a) and (44a) shows that

$$\frac{\partial}{\partial t}\left(\widetilde{\widetilde{\rho u}}\right) - \bar{u}\,\frac{\partial \tilde{\tilde{\rho}}}{\partial t} = \bar{\rho}\,\frac{\partial \tilde{\tilde{u}}}{\partial t}$$

and

$$\frac{\partial}{\partial t}\left(\widetilde{\widetilde{\rho v}}\right) - \bar{v}\,\frac{\partial \tilde{\tilde{\rho}}}{\partial t} = \bar{\rho}\,\frac{\partial \tilde{\tilde{v}}}{\partial t}$$

and the momentum equations (43) and 44) are most conveniently written in the form

$$\bar{\rho}\,\frac{\partial \tilde{\tilde{u}}}{\partial t} + \overline{\rho u}\,\frac{\partial \tilde{\tilde{u}}}{\partial x} + \widetilde{\widetilde{\rho v}}\,\frac{\partial \bar{u}}{\partial y} + \frac{\partial \tilde{\tilde{p}}}{\partial x} = \frac{\partial}{\partial x}\,\tilde{\tilde{\tau}}_{xx}{}^{t} + \frac{\partial}{\partial y}\,\tilde{\tilde{\tau}}_{yx}{}^{t} \tag{43b}$$

$$\bar{\rho}\,\frac{\partial \tilde{\tilde{v}}}{\partial t} + \overline{\rho u}\,\frac{\partial \tilde{\tilde{v}}}{\partial x} + \widetilde{\widetilde{\rho v}}\,\frac{\partial \bar{v}}{\partial y} + \frac{\partial \tilde{\tilde{p}}}{\partial y} = \frac{\partial}{\partial y}\,\tilde{\tilde{\tau}}_{yy}{}^{t} + \frac{\partial}{\partial x}\,\tilde{\tilde{\tau}}_{xy}{}^{t} \tag{44b}$$

where, neglecting products of small terms

$$\widetilde{\widetilde{\rho v}} = \bar{\rho}\tilde{\tilde{v}} + \tilde{\tilde{\rho}}\bar{v} \tag{45}$$

Utilizing equation (45) and a similar result for $\widetilde{\widetilde{\rho u}}$, the convective derivatives in equation (42) may be expanded. The expansion of the latter yields

$$\frac{\partial}{\partial x}\left(\widetilde{\widetilde{\rho u}}\right) = \bar{\rho}\,\frac{\partial \tilde{\tilde{u}}}{\partial x} + \tilde{\tilde{u}}\,\frac{\partial \bar{\rho}}{\partial x} + \tilde{\tilde{\rho}}\,\frac{\partial \bar{u}}{\partial x} + \bar{u}\,\frac{\partial \tilde{\tilde{\rho}}}{\partial x}$$

but the second and third terms on the right are zero by definition of the mean flow so that

$$\frac{\partial}{\partial x}\left(\widetilde{\widetilde{\rho u}}\right) = \bar{\rho}\,\frac{\partial \tilde{\tilde{u}}}{\partial x} + \bar{u}\,\frac{\partial \tilde{\tilde{\rho}}}{\partial x} \tag{46}$$

and likewise

$$\frac{\partial}{\partial y}\left(\widetilde{\widetilde{\rho v}}\right) = \bar{\rho}\,\frac{\partial \tilde{\tilde{v}}}{\partial y} + \tilde{\tilde{v}}\,\frac{\partial \bar{\rho}}{\partial y} + \tilde{\tilde{\rho}}\,\frac{\partial \bar{v}}{\partial y} + \bar{v}\,\frac{\partial \tilde{\tilde{\rho}}}{\partial y} \tag{47}$$

The continuity equation (42) may then be written after substitution of equations (46) and (47) as

$$\frac{\partial \tilde{\tilde{\rho}}}{\partial t} + \bar{u}\,\frac{\partial \tilde{\tilde{\rho}}}{\partial x} + \bar{v}\,\frac{\partial \tilde{\tilde{\rho}}}{\partial y} + \bar{\rho}\left(\frac{\partial \tilde{\tilde{u}}}{\partial x} + \frac{\partial \tilde{\tilde{v}}}{\partial y}\right) + \tilde{\tilde{v}}\,\frac{\partial \bar{\rho}}{\partial y} + \tilde{\tilde{\rho}}\,\frac{\partial \bar{v}}{\partial y} = 0 \tag{48}$$

<u>Perturbation Equations, Final Form</u>

Finally, collecting results, equations (48), (43b), and (44b) provide the governing equations of momentum and continuity in their most tractable form:

$$\frac{\partial \tilde{\tilde{\rho}}}{\partial t} + \bar{u}\,\frac{\partial \tilde{\tilde{\rho}}}{\partial x} + \bar{v}\,\frac{\partial \tilde{\tilde{\rho}}}{\partial y} + \bar{\rho}\left(\frac{\partial \tilde{\tilde{u}}}{\partial x} + \frac{\partial \tilde{\tilde{v}}}{\partial y}\right) + \tilde{\tilde{v}}\,\frac{\partial \bar{\rho}}{\partial y} + \tilde{\tilde{\rho}}\,\frac{\partial \bar{v}}{\partial y} = 0 \tag{49a}$$

$$\bar{\rho}\,\frac{\partial \tilde{\tilde{u}}}{\partial t} + \overline{\rho u}\,\frac{\partial \tilde{\tilde{u}}}{\partial x} + \widetilde{\widetilde{\rho v}}\,\frac{\partial \bar{u}}{\partial y} + \frac{\partial \tilde{\tilde{p}}}{\partial x} = \frac{\partial}{\partial x}\,\tilde{\tilde{\tau}}_{xx}{}^{t} + \frac{\partial}{\partial y}\,\tilde{\tilde{\tau}}_{yx}{}^{t} \tag{49b}$$

$$\bar{\rho}\,\frac{\partial \tilde{\tilde{v}}}{\partial t} + \overline{\rho u}\,\frac{\partial \tilde{\tilde{v}}}{\partial x} + \widetilde{\widetilde{\rho v}}\,\frac{\partial \bar{v}}{\partial y} + \frac{\partial \tilde{\tilde{p}}}{\partial y} = \frac{\partial}{\partial y}\,\tilde{\tilde{\tau}}_{yy}{}^{t} + \frac{\partial}{\partial x}\,\tilde{\tilde{\tau}}_{xy}{}^{t} \tag{49c}$$

with

$$\widetilde{\widetilde{\rho v}} = \bar{\rho}\tilde{\tilde{v}} + \tilde{\tilde{\rho}}\bar{v} \tag{49d}$$

and

$$\left.\begin{aligned}
\tilde{\tilde{\tau}}_{xx}{}^{t} &= 2\mu\,\frac{\partial \tilde{\tilde{u}}}{\partial x} - \frac{2}{3}\,\mu\left(\frac{\partial \tilde{\tilde{u}}}{\partial x} + \frac{\partial \tilde{\tilde{v}}}{\partial y}\right) \\
\tilde{\tilde{\tau}}_{yx}{}^{t} &= \tilde{\tilde{\tau}}_{xy}{}^{t} = \mu\left(\frac{\partial \tilde{\tilde{u}}}{\partial y} + \frac{\partial \tilde{\tilde{v}}}{\partial x}\right) \\
\tilde{\tilde{\tau}}_{yy}{}^{t} &= 2\mu\,\frac{\partial \tilde{\tilde{u}}}{\partial y} - \frac{2}{3}\,\mu\left(\frac{\partial \tilde{\tilde{u}}}{\partial x} + \frac{\partial \tilde{\tilde{v}}}{\partial y}\right)
\end{aligned}\right\} \tag{49e}$$

In formulating these perturbation equations, it has been assumed, as evidenced by equations (49e), that the perturbations do not influence the character of the turbulence, that is:

$$\widetilde{(\rho u)' u'} = 0$$

$$\widetilde{(\rho v)' u'} = 0$$

$$\widetilde{(\rho v)' v'} = 0$$

$$\widetilde{(\rho u)' v'} = 0$$

This assumption, effectively, eliminates the indeterminate situation due to the presence of turbulence and its corresponding "apparent" stress terms and allows one to consider the present problem as a pseudo-laminar flow. Examination of the governing equations, on the other hand, reveals that the mathematical situation consists of three equations in which the "mean" steady flow properties

$$\bar{\rho},\ \bar{u},\ \bar{v},\ \overline{\rho u},\ \mu$$

are regarded as known or prescribed coefficients and the perturbation quantities

$$\overset{\approx}{\rho},\ \overset{\approx}{u},\ \overset{\approx}{v},\ \overset{\approx}{p}$$

are regarded as the dependent variables. Obviously, except in the case of incompressible perturbations, this situation necessitates consideration of the energy equation and an equation of state.

IV. FUTURE WORK

In the next project period the analysis for the unsteady perturbation pressure on stationary wavy walls will be formulated. In addition, an attempt will be made to treat the finite chord length case.

An analysis on the transonic pressure distribution on stationary wavy walls [2] is also anticipated.

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(2) Hosokawa, I., Transonic Flow Past a Wavy Wall, Journal of the Physical Society of Japan, Vol. 15, No. 11, pp. 2080-2086, November 1960.

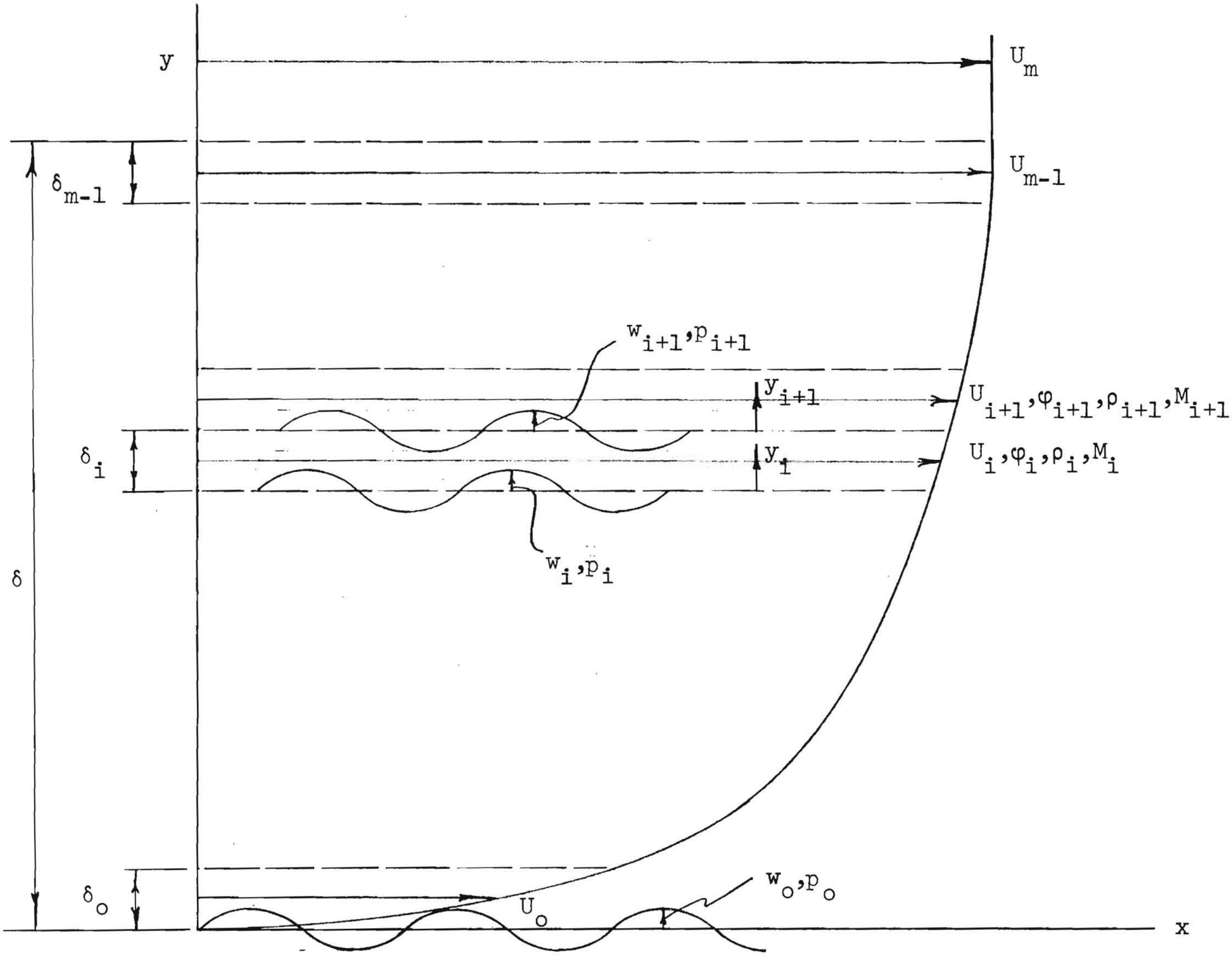

Fig. 1. Idealized Boundary Layer with Notations.

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FINAL REPORT

PROJECT B-208

STUDY OF THE PRESSURE DISTRIBUTION ON OSCILLATING PANELS IN LOW SUPERSONIC FLOW WITH TURBULENT BOUNDARY LAYER

By

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PREFACE

This report covers research initiated by the National Aeronautics and Space Administration, Ames Research Center, Moffett Field, California, under Contract NAS2-2897. The work is administered by Mr. P. A. Gaspers.

The principal investigator of this program is Dr. E. F. E. Zeydel.

This report covers the work done under this contract for the period 11 June 1965 to 10 June 1966.

The author wishes to acknowledge the contribution of Mr. A. C. Bruce for part of the Appendix.

ABSTRACT

A theoretical analysis for the unsteady pressure distribution and generalized aerodynamic force on an oscillating wall of finite extent in the chordwise direction with a thick boundary layer is presented. The analysis is based on a two-dimensional model for the turbulent boundary layer of given velocity profile. Linearized potential flow theory is applied and the finite chord case is treated by using Fourier Integral techniques.

The theoretical equations for a thick, two-dimensional boundary layer with small perturbations are given in the Appendix.

TABLE OF CONTENTS

LIST OF SYMBOLS

A_n = deflection amplitude defined by (7)

a = panel chord

C_n, D_n = defined by (11)

c_n = speed of sound of n^{th} sublayer

E_n, F_n = defined by (14)

$f(\lambda)$ = Fourier Transform of w_0, defined by (57)

$F(\lambda)$ = defined by (51)

G_n = matrix defined by (36)

H = matrix defined by (42)

$H_0^{(2)}$ = Hankel Function of the second kind and zero order

i^2 = - 1

J_0 = Bessel Function of the first kind and zero order

M_n = Mach Number of n^{th} sublayer

P_n = perturbation pressure coefficient defined by (9)

p = static pressure

p_n = perturbation pressure of n^{th} sublayer at $y_n = 0$

q_r = r^{th} generalized coordinate, defined by (60)

Q_r = r^{th} generalized aerodynamic force defined by (63)

R_{rs} = defined by (64)

r, s = integers

T = absolute temperature

t = time

U = local potential flow velocity

U_n	= uniform velocity of n^{th} sublayer
u,v	= x and y velocity components, respectively
w_n	= displacement of n^{th} layer at $y_n = 0$
x,y	= reference coordinate system
Y_0	= Bessel Function of the second kind and zero order
y_n	= reference coordinate of n^{th} sublayer
z_0	= deflection function defined by (55)
α_n	= defined by (31)
$\bar{\alpha}_n$	= defined by (34)
β_n	$= (M_n^2 - 1)^{1/2}$
γ_n	$= (1 - M_n^2)^{1/2}$
δ	= boundary layer thickness
δ_n	= thickness of n^{th} sublayer
$\bar{\delta}_r$	= defined by (70)
ε_r	= defined by (69)
λ	= wave number defined by (1)
μ	= coefficient of viscosity
ρ	= fluid density
ρ_n	= mass air density of n^{th} sublayer
σ_n	= defined by (28)
τ_n	= defined by (29)
τ_{ij}	= stress in j^{th} direction on i^{th} plane
ϕ_n	= velocity potential of n^{th} sublayer
ψ_r	= r^{th} modeshape defined by (59)

Ψ_r = Fourier Transform of ψ_r, defined by (61)

ω = frequency of vibration

Superscripts and others:

t = total, sum of "mean" and fluctuations

' = fluctuation component or differentiation with respect to x

$-$ = mean component

I. INTRODUCTION

In recent years attention has been devoted to the description of the effects of a turbulent boundary layer on the flutter characteristics of aircraft and space vehicle skin panels [1,2]*. In these studies the unsteady pressure distribution on oscillating walls has been estimated either by using a viscous fluid model (McClure) or by using a shear flow model (Fung). Both authors treat only the case of a panel boundary of infinite extent in the chordwise direction.

In this report a theoretical analysis is presented for the unsteady pressure distribution on an oscillating wall of finite extent in the chordwise direction. The analysis is based on a two-dimensional model for the thick boundary layer of given velocity profile. Separating the boundary layer in N sublayers with uniform velocity, the pressure distribution in each sublayer is estimated by applying linearized potential flow theory. The finite chord case is treated by writing the panel boundary in Fourier Integral form and developing the appropriate transfer function for the unsteady pressure distribution. It is shown that the transfer function remains finite for all wavelenghts in the presence of a boundary layer so that well-known procedures can be utilized for numerical evaluation.

The theoretical equations for a thick, two-dimensional boundary layer with small perturbations are given in the Appendix. At present, these equations are not amenable to panel flutter analysis and further development seems warranted only after more detailed information on the shear model is collected.

A brief literature survey is also given in the Appendix.

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*Numbers in square brackets refer to the bibliography.

II. THE PRESSURE DISTRIBUTION ON A TRAVELING WAVY WALL WITH A THICK BOUNDARY LAYER

Consider a two-dimensional laminar flow in the x-direction with a given mean velocity profile U(y). We assume that the mean flow is a boundary layer flow, which joins the external flow, U_N , at $y = \delta$ (see Figure 1).

Let the traveling wavy wall boundary at y = 0 be given by

$$w_0 = A_0 \, e^{i\lambda x} \, e^{i\omega t} \quad ; \quad -\infty < x < +\infty \tag{1}$$

$$-\infty < \lambda < +\infty$$

The problem then becomes to determine the pressure perturbation p_0, at y = 0, and within the boundary layer in terms of w_0.

For simplicity, we divide the boundary layer δ into N sublayers of thicknesses, $\delta_0, \delta_1, \ldots, \delta_n, \ldots, \delta_{N-1}$ and assume that in each sublayer the mean velocity and density is uniform. We also assume that the amplitude A_0 is small compared to the thickness of each sublayer and that the perturbed flows in the sublayer and in the external flow are isentropic and irrotational. The linearized equations for the perturbation velocity potential ϕ can then be utilized to obtain the perturbation velocity and pressure fields within each sublayer. For generality, the external flow velocity U_N is assumed to be either sub- or supersonic so that the mean flow velocity in the sublayers are either sub- or supersonic.

The expressions between the displacements and pressures of a sub- or supersonic sublayer can be obtained as follows.

Let the uniform mean flow velocity, the fluid density, and Mach number of the nth sublayer of thickness, δ_n, be given by U_n, ρ_n, and M_n, respectively (see Figure 1).

The governing equation for the velocity potential ϕ_n in the sublayer is

$$\left(1 - M_n^2\right) \frac{\partial^2 \phi_n}{\partial x^2} + \frac{\partial^2 \phi_n}{\partial y_n^2} - \frac{2M_n}{c_n} \frac{\partial^2 \phi_n}{\partial x \partial t} - \frac{1}{c_n^2} \frac{\partial^2 \phi_n}{\partial t^2} = 0 \tag{2}$$

The boundary conditions for tangential flow are

$$\left.\frac{\partial \phi_n}{\partial y_n}\right|_{y_n = 0} = \frac{\partial w_n}{\partial t} + U_n \frac{\partial w_n}{\partial x} \tag{3}$$

$$\left.\frac{\partial \phi_n}{\partial y_n}\right|_{y_n = \delta_n} = \frac{\partial w_{n+1}}{\partial t} + U_n \frac{\partial w_{n+1}}{\partial x} \tag{4}$$

where w_n and w_{n+1} are the displacements at $y_n = 0$ and $y_n = \delta_n$, respectively.

The perturbation pressures at the edges of the sublayer are given by

$$p_n = - \rho_n \left(\frac{\partial \phi_n}{\partial t} + U_n \frac{\partial \phi_n}{\partial x}\right)\Bigg|_{y_n = 0} \tag{5}$$

$$p_{n+1} = - \rho_n \left(\frac{\partial \phi_n}{\partial t} + U_n \frac{\partial \phi_n}{\partial x}\right)\Bigg|_{y_n = \delta_n} \tag{6}$$

where p_n and p_{n+1} are the pressure perturbations at $y_n = 0$ and $y_n = \delta_n$, respectively.

Let for $-\infty < x < +\infty \quad , \quad -\infty < \lambda < +\infty$

$$w_n = A_n e^{i\lambda x} e^{i\omega t} \tag{7}$$

$$w_{n+1} = A_{n+1} e^{i\lambda x} e^{i\omega t} \tag{8}$$

$$p_n = - \rho_n U_n^2 P_n e^{i\lambda x} e^{i\omega t} \tag{9}$$

$$p_{n+1} = - \rho_{n+1} U_{n+1}^2 P_{n+1} e^{i\lambda x} e^{i\omega t} \tag{10}$$

where A_n, A_{n+1}, P_n and P_{n+1} are in general complex quantities.

If U_n is subsonic, $M_n < 1$ and equation (2) has the solution

$$\phi_n = U_n \left[C_n e^{-\gamma_n \sigma_n y} + D_n e^{+\gamma_n \sigma_n y} \right] e^{i\lambda x} e^{i\omega t} \tag{11}$$

where

$$\sigma_n = \left\{ \left(\lambda - \frac{M_n \omega}{c_n \gamma_n^2} \right)^2 - \frac{\omega^2}{c_n^2 \gamma_n^4} \right\}^{1/2} \tag{12}$$

and

$$\gamma_n = \sqrt{1 - M_n^2} \tag{13}$$

If U_n is supersonic, so that $M_n > 1$, the solution of equation (2) becomes

$$\phi_n = U_n \left[E_n e^{-i\beta_n \tau_n y} + F_n e^{+i\beta_n \tau_n y} \right] e^{i\lambda x} e^{i\omega t} \tag{14}$$

where

$$\tau_n = \left\{ \left(\lambda + \frac{M_n \omega}{c_n \beta_n^2} \right)^2 - \frac{\omega^2}{c_n^2 \beta_n^4} \right\}^{1/2} \tag{15}$$

and

$$\beta_n = \sqrt{M_n^2 - 1} \tag{16}$$

In the region $-\infty < \lambda < \infty$, σ_n and τ_n can be either real or pure imaginary. For the sublayers, where both the positive and negative root is applied, a proper selection of these roots for specific values of λ is not required. For the external flow region, $y > \delta$, however, a selection of the proper root must be based on the radiation and finiteness conditions at infinity.

Let the relative Mach number corresponding to the relative velocity between the traveling wave and the external flow velocity be

$$\bar{M}_N = M_N + \frac{\omega}{\lambda c_N} \tag{17}$$

Solutions of equation (2) with the proper behavior at ∞ then become

a) $\bar{M}_N > 1 \quad ; \quad \lambda > 0$

$$\phi_N = U_N K_N \, e^{i\lambda x} \, e^{-i \left\{\lambda^2\left(\bar{M}_N{}^2 - 1\right)\right\}^{1/2} y} \, e^{i\omega t} \tag{18}$$

b) $\bar{M}_N > 1 \quad ; \quad \lambda > 0$

$$\phi_N = U_N K_N \, e^{i\lambda x} \, e^{+i \left\{\lambda^2\left(\bar{M}_N{}^2 - 1\right)\right\}^{1/2} y} \, e^{i\omega t} \tag{19}$$

c) $-1 \leq \bar{M}_N \leq 1 \quad ; \quad \lambda \gtrless 0$

$$\phi_N = U_N K_N \, e^{i\lambda x} \, e^{- \left\{\lambda^2\left(1 - \bar{M}_N{}^2\right)\right\}^{1/2} y} \, e^{i\omega t} \tag{20}$$

d) $\bar{M}_N < -1 \quad ; \quad \lambda > 0$

$$\phi_N = U_N K_N \, e^{i\lambda x} \, e^{+i \left\{\lambda^2\left(\bar{M}_N{}^2 - 1\right)\right\}^{1/2} y} \, e^{i\omega t} \tag{21}$$

e) $\bar{M}_N < -1 \quad ; \quad \lambda < 0$

$$\phi_N = U_N K_N \, e^{i\lambda x} \, e^{-i \left\{\lambda^2\left(\bar{M}_N{}^2 - 1\right)\right\}^{1/2} y} \, e^{i\omega t} \tag{22}$$

Interpreting (18) - (22) in the notation of (11) and (14), using only the first term in the square bracket, we find after a few algebraic manipulations, that the solution of equation (2) for the external flow region can be written as

a) For $M_N < 1$,

$$\phi_N = U_N C_N \, e^{-\gamma_N \sigma_N y} \, e^{i\lambda x} \, e^{i\omega t} \tag{23}$$

where

$$\sigma_N = \left\{\left(\lambda - \frac{M_N \omega}{c_N \gamma_N^2}\right)^2 - \frac{\omega^2}{c_N^2 \gamma_N^4}\right\}^{1/2} \quad ; \quad \lambda > \frac{\omega}{c_N(1 - M_N)}$$

$$\text{or } \lambda < -\frac{\omega}{c_N(1 + M_N)}$$

$$= i \left\{\frac{\omega^2}{c_N^2 \gamma_N^4} - \left(\lambda - \frac{\omega^2}{c_N \gamma_N^2}\right)^2\right\}^{1/2} \quad ; \quad -\frac{\omega}{c_N(1 + M_N)} < \lambda < \frac{\omega}{c_N(1 - M_N)}$$

(24)

and

$$\gamma_N = \sqrt{1 - M_N^2} \tag{25}$$

b) For $M_N > 1$,

$$\phi_N = U_N E_N \, e^{-i\beta_N \tau_N y} \, e^{i\lambda x} \, e^{i\omega t} \tag{26}$$

where

$$\tau_N = \left\{\left(\lambda + \frac{M_N\omega}{c_N\beta_N^2}\right)^2 - \frac{\omega^2}{c_N^2\beta_N^4}\right\}^{1/2} \quad ; \quad \lambda > -\frac{\omega}{c_N(M_N+1)}$$

$$= -i\left\{\frac{\omega^2}{c_N^2\beta_N^4} - \left(\lambda + \frac{M_N\omega}{c_N\beta_N^2}\right)^2\right\}^{1/2} \quad ; \quad -\frac{\omega}{c_N(M_N-1)} < \lambda < -\frac{\omega}{c_N(M_N+1)}$$

$$= -\left\{\left(\lambda + \frac{M_N\omega}{c_N\beta_N^2}\right)^2 - \frac{\omega^2}{c_N^2\beta_N^4}\right\}^{1/2} \quad ; \quad \lambda < -\frac{\omega}{c_N(M_N-1)}$$

$$\beta_N = \sqrt{M_N^2 - 1} \tag{27}$$

Although strictly speaking not required, it is convenient to define, in general

$$\sigma_n = \left\{\left(\lambda - \frac{M_n\omega}{c_n\gamma_n^2}\right)^2 - \frac{\omega^2}{c_n^2\gamma_n^4}\right\}^{1/2} \quad ; \quad \lambda > \frac{\omega}{c_n(1-M_n)}$$

$$\text{or} \quad \lambda < -\frac{\omega}{c_n(1+M_n)}$$

$$= i\left\{\frac{\omega}{c_n^2\gamma_n^4} - \left(\lambda - \frac{M_n\omega}{c_n\gamma_n^2}\right)^2\right\}^{1/2} \quad ; \quad -\frac{\omega}{c_n(1+M_n)} < \lambda < \frac{\omega}{c_n(1-M_n)}$$

$$\tag{28}$$

and

$$\tau_n = \left\{\left(\lambda + \frac{M_n\omega}{c_n\beta_n^2}\right)^2 - \frac{\omega^2}{c_n^2\beta_n^4}\right\}^{1/2} \quad ; \quad \lambda > -\frac{\omega}{c_n(M_n+1)}$$

$$= -i\left\{\frac{\omega^2}{c_n^2\beta_n^4} - \left(\lambda + \frac{M_n\omega}{c_n\beta_n^2}\right)^2\right\}^{1/2} \quad ; \quad -\frac{\omega}{c_n(M_n-1)} < \lambda < -\frac{\omega}{c_n(M_n+1)}$$

$$= -\left\{\left(\lambda + \frac{M_n\omega}{c_n\beta_n^2}\right)^2 - \frac{\omega^2}{c_n^2\beta_n^4}\right\}^{1/2} \quad ; \quad \lambda < -\frac{\omega}{c_n(M_n-1)} \qquad (29)$$

for the sublayers as well as the external flow region.

The relation between the deflection and pressure coéfficients for the sublayers can be obtained as follows.

If U_n is subsonic, $M_n < 1$, we eliminate C_n and D_n, using (3) - (5), (7) - (9) and (11) and find

$$A_{n+1} = \frac{e^{-\alpha_n} + e^{+\alpha_n}}{2} A_n + \frac{\gamma_n\sigma_n}{\lambda^2} \frac{U_n^2}{\left(\frac{\omega}{\lambda} + U_n\right)^2} \frac{e^{-\alpha_n} - e^{+\alpha_n}}{2} P_n \qquad (30)$$

where

$$\alpha_n = \gamma_n\sigma_n\delta_n \qquad (31)$$

Similarly, using (3), (5) - (7), (9) - (11), yields

$$P_{n+1} = \frac{\rho_n U_n^2}{\rho_{n+1} U_{n+1}^2} \frac{\left(\frac{\omega}{\lambda} + U_n\right)^2}{U_n^2} \frac{\lambda^2}{\gamma_n \sigma_n} \frac{e^{-\alpha_n} - e^{+\alpha_n}}{2} A_n$$

$$+ \frac{\rho_n U_n^2}{\rho_{n+1} U_{n+1}^2} \frac{e^{-\alpha_n} + e^{+\alpha_n}}{2} P_n \tag{32}$$

If U_n is supersonic, $M_n > 1$, we find using (3) - (5), (7) - (9) and (14),

$$A_{n+1} = \frac{e^{-\bar{\alpha}_n} + e^{+\bar{\alpha}_n}}{2} A_n + \frac{i\beta_n \tau_n}{\lambda^2} \frac{U_n^2}{\left(\frac{\omega}{\lambda} + U_n\right)^2} \frac{e^{-\bar{\alpha}_n} - e^{+\bar{\alpha}_n}}{2} P_n \tag{33}$$

where

$$\bar{\alpha}_n = i\beta_n \tau_n \delta_n \tag{34}$$

and using (3), (5) - (7), (9) - (11), and (14)

$$P_{n+1} = \frac{\rho_n U_n^2}{\rho_{n+1} U_{n+1}^2} \frac{\left(\frac{\omega}{\lambda} + U_n\right)^2}{U_n^2} \frac{\lambda^2}{i\beta_n \tau_n} \frac{e^{-\bar{\alpha}_n} - e^{+\bar{\alpha}_n}}{2} A_n$$

$$+ \frac{\rho_n U_n^2}{\rho_{n+1} U_{n+1}^2} \frac{e^{-\bar{\alpha}_n} + e^{+\bar{\alpha}_n}}{2} P_n \tag{35}$$

For clarity, we write (30), (32), (33) and (35) in matrix form

$$\begin{vmatrix} A_{n+1} \\ \\ P_{n+1} \end{vmatrix} = \begin{bmatrix} G_n \end{bmatrix} \begin{vmatrix} A_n \\ \\ P_n \end{vmatrix} \tag{36}$$

where the elements of G_n are given by , for $M_n \leq 1$,

$$g_{n_{11}} = \frac{e^{-\alpha_n} + e^{+\alpha_n}}{2}$$

$$g_{n_{12}} = \frac{\gamma_n \sigma_n}{\lambda^2} \frac{U_n^2}{\left(\frac{\omega}{\lambda} + U_n\right)^2} \frac{e^{-\alpha_n} - e^{+\alpha_n}}{2}$$

$$g_{n_{21}} = \frac{\rho_n U_n^2}{\rho_{n+1} U_{n+1}^2} \frac{\left(\frac{\omega}{\lambda} + U_n\right)^2}{U_n^2} \frac{\lambda^2}{\gamma_n \sigma_n} \frac{e^{-\alpha_n} - e^{+\alpha_n}}{2}$$

$$g_{n_{22}} = \frac{\rho_n U_n^2}{\rho_{n+1} U_{n+1}^2} \frac{e^{-\alpha_n} + e^{+\alpha_n}}{2} \tag{37}$$

where

$$\alpha = \gamma_n \sigma_n \delta_n$$

$$\gamma_n = \sqrt{1 - M_n^2}$$

σ_n is defined by (28) (38)

and, for $M_n > 1$,

$$g_{n_{11}} = \frac{e^{-\bar{\alpha}_n} + e^{+\bar{\alpha}_n}}{2}$$

$$g_{n_{12}} = \frac{i\beta_n\tau_n}{\lambda^2} \frac{U_n^{\,2}}{\left(\frac{\omega}{\lambda} + U_n\right)^2} \frac{e^{-\bar{\alpha}_n} - e^{+\bar{\alpha}_n}}{2}$$

$$g_{n_{21}} = \frac{\rho_n U_n^{\,2}}{\rho_{n+1}\, U_{n+1}^{\,2}} \frac{\left(\frac{\omega}{\lambda} + U_n\right)^2}{U_n^{\,2}} \frac{\lambda^2}{i\beta_n\tau_n} \frac{e^{-\bar{\alpha}_n} - e^{+\bar{\alpha}_n}}{2}$$

$$g_{n_{22}} = \frac{\rho_n U_n^{\,2}}{\rho_{n+1}\, U_{n+1}^{\,2}} \frac{e^{-\bar{\alpha}_n} + e^{+\bar{\alpha}_n}}{2} \tag{39}$$

where

$$\bar{\alpha}_n = i\beta_n\tau_n\delta_n$$

$$\beta_n = \sqrt{M_n^{\,2} - 1}$$

τ_n is defined by (29) (40)

Consequently,

$$\begin{vmatrix} A_N \\ \\ P_N \end{vmatrix} = \left[H\right] \begin{vmatrix} A_0 \\ \\ P_0 \end{vmatrix} \tag{41}$$

where

$$\left[H\right] = \left[G_{N-1}\right] \left[G_{N-2}\right] \cdot \cdot \cdot \left[G_0\right] \tag{42}$$

If the flow external to the boundary layer is subsonic, so that $M_n < 1$, its velocity potential is given by (23) and we find, using (3) and (7),

$$C_N = - \frac{i\left(\frac{\omega}{\lambda} + U_N\right)\lambda}{\gamma_N \sigma_N U_N} A_N \tag{43}$$

so that from (5) and (9) there follows

$$P_N = \frac{\left(\frac{\omega}{\lambda} + U_N\right)^2 \lambda^2}{\gamma_N \sigma_N U_N{}^2} A_N \tag{44}$$

The pressure coefficient, P_0, is obtained from (41) and (44)

$$P_0 = - \frac{h_{21} - \dfrac{\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\gamma_N \sigma_N U_N{}^2} h_{11}}{h_{22} - \dfrac{\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\gamma_N \sigma_N U_N{}^2} h_{12}} A_0 \quad ; \quad M_N < 1 \tag{45}$$

where h_{11}, h_{12}, h_{21}, and h_{22} are the elements of the matrix H.

If the external flow is supersonic, $M_N > 1$, its velocity potential is given by (26) and there follows from (3) and (7),

$$E_N = - \frac{\left(\frac{\omega}{\lambda} + U_N\right)\lambda}{\beta_N \tau_N U_N} A_N \tag{46}$$

and from (5) and (9)

$$P_N = -\frac{i\left(\frac{\omega}{\lambda} + U_N\right)^2 \lambda^2}{\beta_N \tau_N U_N^{\,2}} A_N \tag{47}$$

Using (41) and (44), the pressure coefficient, P_0, becomes

$$P_0 = -\frac{h_{21} + \dfrac{i\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\beta_N \tau_N\ U_N^{\,2}}\ h_{11}}{h_{22} + \dfrac{i\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\beta_N \tau_N\ U_N^{\,2}}\ h_{12}}\ A_0 \quad ; \quad M_N > 1 \tag{48}$$

The perturbation pressure at the surface of the traveling wavy wall

$$w_0 = A_0\ e^{i\lambda x}\ e^{i\omega t} \quad ; \quad \begin{array}{l} -\infty < x < +\infty \\ -\infty < \lambda < +\infty \end{array} \tag{49}$$

is given by

$$p_0 = -\rho_0 U_0^{\,2} P_0\ e^{i\lambda x}\ e^{i\omega t}$$

or

$$p_0 = -\rho_N U_N^{\,2}\ F(\lambda) A_0\ e^{i\lambda x}\ e^{i\omega t} \tag{50}$$

where

$$F(\lambda) = -\frac{\rho_0 U_0^2}{\rho_N U_N^2} \; \frac{h_{21} - \dfrac{\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\gamma_N \sigma_N \; U_N^2} \; h_{11}}{h_{22} - \dfrac{\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\gamma_N \sigma_N \; U_N^2} \; h_{12}} \quad ; \quad M_N < 1$$

$$= -\frac{\rho_0 U_0^2}{\rho_N U_N^2} \; \frac{h_{21} + \dfrac{i\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\beta_N \tau_N \; U_N^2} \; h_{11}}{h_{22} + \dfrac{i\lambda^2\left(\frac{\omega}{\lambda} + U_N\right)^2}{\beta_N \tau_N \; U_N^2} \; h_{12}} \quad ; \quad M_N > 1 \qquad (51)$$

It is interesting to note the following characteristics of the function $F(\lambda)$,

(a) If $\delta \to 0$, $\delta_n \to 0$ and

$$g_{n_{11}} = 1 \quad ; \quad g_{n_{12}} = g_{n_{21}} = 0$$

$$g_{n_{22}} = \frac{\rho_n U_n^2}{\rho_{n+1} \; U_{n+1}^2}$$

Thus,

$$h_{11} = 1 \quad ; \quad h_{12} = h_{21} = 0$$

$$h_{22} = \frac{\rho_0 U_0^2}{\rho_N U_N^2}$$

and

$$F(\lambda) = + \frac{\left(\omega + U_N\lambda\right)^2}{\gamma_N\sigma_N\ U_N^{\ 2}} \quad ; \quad M_N < 1$$

$$= - \frac{i\left(\omega + U_N\lambda\right)^2}{\beta_N\tau_N\ U_N^{\ 2}} \quad ; \quad M_N > 1 \qquad (52)$$

so that

$$P_0 = - \frac{\rho_N\left(\omega + U_N\lambda\right)^2}{\gamma_N\sigma_N} A_0\ e^{i\lambda x}\ e^{i\omega t} \quad ; \quad M_N < 1$$

$$= + \frac{i\rho_N\left(\omega + U_N\lambda\right)^2}{\beta_N\tau_N} A_0\ e^{i\lambda x}\ e^{i\omega t} \quad ; \quad M_N > 1 \qquad (53)$$

which is the expected traveling wavy wall solution.

(b) If $\delta = 0$ and $\omega = 0$, we obtain the stationary wavy wall solution without a boundary layer

$$P_0 = - \frac{\rho_N U_N^{\ 2}|\lambda|}{\gamma_N} A_0\ e^{i\lambda x} \quad ; \quad M_N < 1$$

$$= \frac{i\rho_N U_N^{\ 2}\lambda}{\beta_N} A_0\ e^{i\lambda x} \quad ; \quad M_N > 1 \qquad (54)$$

(c) If $\delta \neq 0$ and $\omega = 0$ the results correspond to those given in progress report No. 2 for a stationary wavy wall with a thick boundary layer.

Of importance for the numerical evaluation of $F(\lambda)$ is the fact that only if $\delta = 0$ and $\omega \neq 0$, $F(\lambda)$ contains singular points, since σ_N and τ_N become zero for specific values of λ. However, if $\delta \neq 0$, $F(\lambda)$ remains finite for all values of λ.

III. THE UNSTEADY PRESSURE DISTRIBUTION AND GENERALIZED FORCE ON AN OSCILLATING WALL OF FINITE CHORD LENGTH WITH A THICK BOUNDARY LAYER

The unsteady pressure distribution on an oscillating wall of finite chord length with a thick boundary layer can be obtained from the previous results by writing the wall boundary in Fourier integral form.

Let the oscillating wall boundary at y = 0 be given by

$$w_0(x,t) = z_0(x)\, e^{i\omega t} \quad ; \quad 0 \le x \le a$$

$$= 0 \quad ; \quad x < 0 \text{ or } x > a \tag{55}$$

In Fourier integral form, (51) becomes

$$w_0(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\lambda)\, e^{i\lambda x}\, d\lambda\, e^{i\omega t} \tag{56}$$

where

$$f(\lambda) = \int_{-\infty}^{+\infty} z_0(x)\, e^{-i\lambda x}\, dx = \int_0^a z_0(x)\, e^{-i\lambda x}\, dx \tag{57}$$

The unsteady pressure distributions at y = 0 follows directly from (49), (50), and (56)

$$p_0(x,t) = -\frac{\rho_N U_N^{\,2}}{2\pi}\int_{-\infty}^{+\infty} F(\lambda)\, f(\lambda)\, e^{i\lambda x}\, d\lambda\, e^{i\omega t} \tag{58}$$

It is customary in panel flutter analysis to utilize a Ritz-Galerkin method to obtain the flutter boundaries and to satisfy the panel boundary conditions by choosing a suitable set of deflection functions. Therefore, let

$$z_0(x) = \sum_r q_r \psi_r(x) \tag{59}$$

where $\Psi_r(x)$ satisfies the appropriate boundary conditions. From (57)

$$f(\lambda) = \sum_r q_r \Psi_r(x) \tag{60}$$

where

$$\Psi_r(\lambda) = \int_0^a \psi_r(x)\, e^{-i\lambda x}\, dx \tag{61}$$

The unsteady pressure distribution at y = 0 becomes

$$p_0(x,t) = -\frac{\rho_N U_N^2}{2\pi} \sum_r q_r \int_{-\infty}^{+\infty} F(\lambda)\, \Psi_r(\lambda)\, e^{i\lambda x}\, d\lambda\, e^{i\omega t} \tag{62}$$

The generalized aerodynamic force is defined by

$$Q_r(t) = \int_0^a P_0(x,t)\, \psi_r(x)\, dx$$

$$= -\frac{\rho_N U_N^2}{2\pi} \sum_s q_s R_{rs} e^{i\omega t} \tag{63}$$

where

$$R_{rs} = \int_{-\infty}^{+\infty} F(\lambda)\, \Psi_r(-\lambda)\, \Psi_s(\lambda)\, d\lambda \tag{64}$$

For a panel with primed edges at the leading and trailing edges $(x = 0, a)$,

$$\psi_r(x) = \sin \frac{r\pi}{a} x \tag{65}$$

and

$$\Psi_r(\lambda) = \int_0^a \sin \frac{r\pi}{a} x \, e^{-i\lambda x} \, dx = \frac{r\frac{\pi}{a}}{\left(r\frac{\pi}{a}\right)^2 - \lambda^2} \left[1 - (-1)^r e^{-i\lambda a}\right] \tag{66}$$

Thus,

$$R_{rs} = \frac{rs\pi^2}{a^2} \int_{-\infty}^{+\infty} F(\lambda) \frac{\{1 - (-1)^r e^{+i\lambda a}\}\{1 - (-1)^s e^{-i\lambda a}\}}{\{\left(r\frac{\pi}{a}\right)^2 - \lambda^2\}\{\left(s\frac{\pi}{a}\right)^2 - \lambda^2\}} d\lambda \tag{67}$$

where $F(\lambda)$ is defined by (51).

For a panel with clamped leading and trailing edges

$$\Psi_r(x) = \mathrm{Cosh}\ \bar{\delta}_r x - \cos \bar{\delta}_r x - \varepsilon_r (\mathrm{Sinh}\ \bar{\delta}_r x - \sin \bar{\delta}_r x) \tag{68}$$

where

$$\varepsilon_r = \frac{\mathrm{Cosh}\ \bar{\delta}_r a - \cos \bar{\delta}_r a}{\mathrm{Sinh}\ \bar{\delta}_r a - \sin \bar{\delta}_r a} \tag{69}$$

and δ_r is given by the characteristic equation

$$\mathrm{Cosh}\ \bar{\delta}_r a \cos \bar{\delta}_r a = 1 \tag{70}$$

We write $\psi_r(x)$ in the form

$$\psi_r(x) = \tfrac{1}{2}(1 - \varepsilon_r)\, e^{\bar{\delta}_r x} + \tfrac{1}{2}(1 + \varepsilon_r)\, e^{-\bar{\delta}_r x}$$

$$- \tfrac{1}{2}(1 + \varepsilon_r)\, e^{+i\bar{\delta}_r x} - \tfrac{1}{2}(1 - \varepsilon_r)\, e^{-i\bar{\delta}_r x} \qquad (71)$$

Since

$$\int_0^a e^{\alpha x}\, e^{-i\lambda x}\, dx = \frac{1}{\alpha - i\lambda}\left[e^{(\alpha - i\lambda)a} - 1\right] , \qquad (72)$$

$$\Psi_r(\lambda) = \int_0^a \psi_r(\lambda)\, e^{-i\lambda x}\, dx = \frac{(1 - \varepsilon_r)}{2(\bar{\delta}_r - i\lambda)}\left[e^{(\bar{\delta}_r - i\lambda)a} - 1\right]$$

$$- \frac{(1 + \varepsilon_r)}{2(\bar{\delta}_r + i\lambda)}\left[e^{-(\bar{\delta}_r + i\lambda)a} - 1\right] + \frac{i(1 + \varepsilon_r)}{2(\bar{\delta}_r - \lambda)}\left[e^{i(\bar{\delta}_r - \lambda)a} - 1\right]$$

$$- \frac{i(1 - \varepsilon_r)}{2(\bar{\delta}_r + \lambda)}\left[e^{-i(\bar{\delta}_r + \lambda)a} - 1\right] \qquad (73)$$

Using (73), R_{rs} for the clamped edge case follows again from (64).

The main difficulty in the foregoing analysis will be the evaluation of the infinite integral in the expression for R_{rs}. Since it is to be expected that $F(\lambda)$ as well as $\Psi_r(\lambda)$ are oscillatory with λ special care should be exercised to maintain accuracy if numerical techniques are applied. A cursory analysis, in which the effects of the boundary layer are ignored, shows that $F(\lambda)$ increases linearly with λ. If it is assumed that also with a thick boundary layer $F(\lambda)$ is proportional to λ, the kernel of the infinite integral in the expression for R_{rs} is at least proportional to $1/\lambda$, since $\Psi_r(\lambda)$ is at least proportional to $1/\lambda$.

In view of these circumstances it is proposed in the continuation of these research efforts to numerically evaluate the function R_{rs} for typical boundary layer profiles and panel design configurations to obtain a better understanding of the difficulties involved. Should the analysis prove promising, the effects of a thick boundary layer on the flutter characteristics of a two-dimensional panel configuration should be studied, followed by an extension of the present theories to the three-dimensional case.

IV. RECOMMENDATIONS AND FUTURE WORK

It is recommended to extend the theoretical developments presented in this report along the following lines:

1) Numerically evaluate the function R_{rs} for typical boundary layer profiles and panel configurations to obtain a better understanding of the general behavior of the function.

2) Numerically evaluate the pressure distribution on a finite stationary wavy wall and compare the results with available experimental data.

3) Conduct a flutter analysis using the two-dimensional aerodynamic theories developed and compare the results with available experimental data.

4) Extend the present theories to the three-dimensional case.

APPENDIX A

UNSTEADY POTENTIAL FLOW EQUATIONS

It is interesting to develop from the previous results the more familiar forms of the potential flow equations without boundary layer.

Unsteady Subsonic Flow ($\delta = 0$)

We have seen that for a traveling wavy wall boundary at y = 0 given by

$$w_0 = A_0\, e^{i\lambda x}\, e^{i\omega t} \quad ; \quad -\infty < x < +\infty \tag{A-1}$$

the subsonic potential becomes [see (23) and (43)]

$$\phi_N = -\frac{i(\omega + \lambda U_N)}{\gamma_N \sigma_N} A_0\, e^{-\gamma_N \sigma_N y}\, e^{i\lambda x}\, e^{i\omega t} \tag{A-2}$$

where σ_N and γ_N are given by (24) and (25), respectively.

On the other hand, when the oscillating wall boundary at y = 0 is given by [see (55)]

$$\begin{aligned} w_0(x,t) &= z_0(x)\, e^{i\omega t} \quad &;& \quad 0 \leq x \leq a \\ &= 0 \quad &;& \quad x < 0 \text{ or } x > a \end{aligned} \tag{A-3}$$

we find using (56) and (57) that

$$\phi_N = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i(\omega + \lambda U_N)}{\gamma_N \sigma_N}\, e^{-\gamma_N \sigma_N y}\, f(\lambda)\, e^{i\lambda x}\, d\lambda\, e^{i\omega t} \tag{A-4}$$

where $f(\lambda)$, given by (56), is the Fourier Transform of w_0.

Since

$$i\{\omega + \lambda U_N\}f(\lambda) \quad e^{i\omega t} = \text{F.T.}\left(\frac{\partial w_0}{\partial t} + U_N \frac{\partial w_0}{\partial x}\right) \tag{A-5}$$

there follows from (A-4)

$$\text{F.T.}(\phi_N) = - \text{F.T.}\left(\frac{\partial w_0}{\partial t} + U_N \frac{\partial w_0}{\partial x}\right) \text{F.T.}(Q) \tag{A-6}$$

where

$$\text{F.T.}(Q) = \frac{e^{-\gamma_N \sigma_N y}}{\gamma_N \sigma_N} \tag{A-7}$$

Using (24) we find that

$$\text{F.T.}\left\{ e^{-i \frac{M_N \omega}{c_N \gamma_N^2} x} Q \right\} = \frac{e^{-\gamma_N \bar{\sigma}_N y}}{\gamma_N \bar{\sigma}_N} \tag{A-8}$$

where

$$\bar{\sigma}_N = \left(\lambda^2 - \frac{\omega^2}{c_N^2 \gamma_N^4}\right)^{1/2} \quad ; \quad \lambda > \frac{\omega}{c_N \gamma_N^2} \text{ or } \lambda < - \frac{\omega}{c_N \gamma_N^2}$$

$$= i \left(\frac{\omega^2}{c_N^2 \gamma_N^4} - \lambda^2\right)^{1/2} \quad ; \quad - \frac{\omega}{c_N \gamma_N^2} < \lambda < \frac{\omega}{c_N \gamma_N^2} \tag{A-9}$$

Thus,

$$e^{-i\frac{M_N\omega}{c_N\gamma_N^2}x}\,Q = \frac{1}{\pi\gamma_N}\int_{\frac{\omega}{c_N\gamma_N^2}}^{\infty}\frac{e^{-\gamma_N\left(\lambda^2-\frac{\omega^2}{c_N^2\gamma_N^4}\right)^{1/2}y}}{\left(\lambda^2-\frac{\omega^2}{c_N^2\gamma_N^4}\right)^{1/2}}\cos\lambda x\,d\lambda$$

$$-\frac{i}{\pi\gamma_N}\int_0^{\frac{\omega}{c_N\gamma_N^2}}\frac{\cos\left\{\gamma_N\left(\frac{\omega^2}{c_N^2\gamma_N^4}-\lambda^2\right)^{1/2}\right\}}{\left(\frac{\omega^2}{c_N^2\gamma_N^4}-\lambda^2\right)^{1/2}}\cos\lambda x\,d\lambda$$

$$-\frac{1}{\pi\gamma_N}\int_0^{\frac{\omega}{c_N\gamma_N^2}}\frac{\sin\left\{\gamma_N\left(\frac{\omega^2}{c_N^2\gamma_N^4}-\lambda^2\right)^{1/2}\right\}}{\left(\frac{\omega^2}{c_N^2\gamma_N^4}-\lambda^2\right)^{1/2}}\cos\lambda x\,d\lambda \tag{A-10}$$

Now [5 , p. 30]

$$\int_0^{\infty} g_1(x)\cos(xy)\,dx = \frac{\pi}{2}\,Y_0\left[a(b^2+y^2)^{1/2}\right]$$

where

$$g_1(x) = (a^2 - x^2)^{-1/2} \sin\left[b(a^2 - x^2)^{1/2}\right] \quad ; \quad 0 < x < a$$

$$= -(x^2 - a^2)^{-1/2} e^{-b(x^2 - a^2)^{1/2}} \quad ; \quad x > a$$

and

$$\int_0^\infty g_2(x) \cos(xy)\, dx = \frac{\pi}{2} J_0\left[a(b^2 + y^2)^{1/2}\right]$$

where

$$g_2(x) = (a^2 - x^2)^{-1/2} \cos\left[b(a^2 - x^2)^{1/2}\right] \quad ; \quad 0 < x < a$$

$$= 0 \quad ; \quad x > a$$

Thus,

$$Q = -\frac{1}{2\gamma_N} e^{i \frac{M\omega}{c_N \gamma_N^2} x} H_0^{(2)}\left[\frac{\omega}{c_N \gamma_N^2}(x^2 + \gamma_N^2 y^2)^{1/2}\right] \qquad \text{(A-11)}$$

Taking the inverse transform of (A-5) and using (A-11), we find the well-known result [3]

$$\phi = \frac{i}{2\gamma_N} \int_0^a \left(\frac{\partial w_0(\xi, t)}{\partial t} + U_N \frac{\partial w_0(\xi, t)}{\partial \xi}\right) e^{+i \frac{M_N \omega}{c_N \gamma_N^2}(x - \xi)}$$

$$\times H_0^{(2)}\left\{\frac{\omega}{c_N \gamma_N^2}\left[(x - \xi)^2 + y^2\right]^{1/2}\right\} d\xi \qquad \text{(A-12)}$$

If $\omega = 0$, (A-9) gives

$$\bar{\sigma}_N = |\lambda| \quad ; \quad -\infty < \lambda < +\infty$$

and

$$Q = \frac{1}{\pi\gamma_N} \int_0^{\infty} \frac{e^{-\gamma_N \lambda y}}{\gamma} \cos \lambda x \, d\lambda$$

$$= -\frac{1}{2\pi\gamma_N} \ln \left[x^2 + \gamma_N{}^2 y^2 \right] \tag{A-13}$$

thus,

$$\phi_N = + \frac{1}{2\pi\gamma_N} \int_0^a U_N \frac{\partial w_0(\xi,t)}{\partial \xi} \ln \left[(x-\xi)^2 + \gamma_N{}^2 y^2 \right] d\xi \tag{A-14}$$

{See [4 , p. 213] }.

<u>Unsteady Supersonic Flow ($\delta = 0$)</u>

The supersonic case can be obtained in a similar way using the supersonic potential corresponding to (A-1) { see (26) and (46) }

$$\phi_N = - \frac{(\omega + \lambda U_N)}{\beta_N \tau_N} A_0 \, e^{-i\beta_N \tau_N y} \, e^{i\lambda x} \, e^{i\omega t} \tag{A-15}$$

The velocity potential corresponding to (A-3) becomes

$$\phi_N = - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{(\omega + \lambda U_N)}{\beta_N \tau_N} e^{-i\beta_N \tau_N y} f(\lambda) \, e^{i\lambda x} \, d\lambda \, e^{i\omega t} \tag{A-16}$$

Using (A-5), there follows

$$\text{F.T.}(\phi_N) = \text{F.T.} \left(\frac{\partial w_0}{\partial t} + U_N \frac{\partial w_N}{\partial t} \right) \text{F.T.}\{\bar{Q}\} \tag{A-17}$$

where

$$\text{F.T.}(\bar{Q}) = \frac{ie^{-i\beta_N \tau_N y}}{\beta_N \tau_N} \tag{A-18}$$

so that, using (27)

$$\text{F.T.} \left\{ e^{+i \frac{M_N \omega}{c_N \beta_N^2} x} \bar{Q} \right\} = \frac{ie^{-i\beta_N \tau_N y}}{\beta_N \bar{\tau}_N} \tag{A-19}$$

where

$$\bar{\tau}_N = \left(\lambda^2 - \frac{\omega^2}{c_N^2 \beta_N^4} \right)^{1/2} \quad ; \quad \lambda > \frac{\omega}{c_N \beta_N^2}$$

$$= - i \left(\frac{\omega^2}{c_N^2 \beta_N^4} - \lambda^2 \right)^{1/2} \quad ; \quad - \frac{\omega}{c_N \beta_N^2} < \lambda < \frac{\omega}{c_N{}_N^2}$$

$$= - \left(\lambda^2 - \frac{\omega^2}{c_N^2 \beta_N^4} \right)^{1/2} \quad ; \quad \lambda < - \frac{\omega}{c_N \beta_N^2} \tag{A-20}$$

Consequently,

$$e^{+i\frac{M_N\omega}{c_N\beta_N^2}x}\,\bar{Q} = -\frac{1}{\pi\beta_N}\int_0^{\frac{\omega}{c_N\beta_N^2}} \frac{e^{-\beta_N\left(\frac{\omega^2}{c_N^2\beta_N^4}-\lambda^2\right)^{1/2} y}}{\left(\frac{\omega^2}{c_N^2\beta_N^4}-\lambda^2\right)^{1/2}} \cos\lambda x \, d\lambda$$

$$-\frac{1}{\pi\beta_N}\int_{\frac{\omega}{c_N\beta_N^2}}^{\infty} \frac{\cos\left\{\beta_N\left(\lambda^2-\frac{\omega^2}{c_N^2\beta_N^4}\right)^{1/2}\right\}}{\left(\lambda^2-\frac{\omega^2}{c_N^2\beta_N^4}\right)^{1/2}} \sin\lambda x \, d\lambda$$

$$+\frac{1}{\pi\beta_N}\int_{\frac{\omega}{c_N\beta_N^2}}^{\infty} \frac{\sin\left\{\beta_N\left(\lambda^2-\frac{\omega^2}{c_N^2\beta_N^4}\right)^{1/2}\right\}}{\left(\lambda^2-\frac{\omega^2}{c_N^2\beta_N^4}\right)^{1/2}} \cos\lambda x \, d\lambda \qquad \text{(A-21)}$$

Now, [5, p. 30]

$$\int_0^{\infty} g_3(x)\cos(xy)\,dx = \frac{\pi}{2} J_0\left[a(y^2-b^2)^{1/2}\right] \quad ; \quad y > b$$

$$= 0 \quad ; \quad y < b$$

where

$$g_3(x) = (a^2 - x^2)^{-1/2} e^{-b(a^2 - x^2)^{1/2}} \quad ; \quad 0 < x < a$$

$$= - (x^2 - a^2)^{-1/2} \sin\left[b(a^2 - x^2)^{1/2}\right] \quad ; \quad x > a$$

and [5, p. 142]

$$\int_0^\infty g_4(x) \sin(xy)\, dx = \frac{\pi}{2} J_0\left[a(y^2 - b^2)^{1/2}\right] \quad ; \quad y > b$$

$$= 0 \quad ; \quad y < b$$

where

$$g_4(x) = 0 \quad ; \quad 0 < x < a$$

$$= (x^2 - a^2)^{-1/2} \cos\left[b(x^2 - a^2)^{1/2}\right] \quad ; \quad x > a$$

Thus,

$$Q = -\frac{1}{\beta_N} e^{i \frac{M_N \omega}{c_N \beta_N{}^2} x} J_0\left\{\frac{\omega}{c_N \beta_N{}^2}\left(x^2 - \beta_N{}^2 y^2\right)^{1/2}\right\} ; \quad x > \beta_N y$$

$$= 0 \quad ; \quad x < \beta_N y$$

(A-22)

Taking the inverse transform of (A-17) yields {see [3] }

$$\phi_N = -\frac{1}{\beta_N}\int_0^{x-\beta_N y}\left\{\frac{\partial w_0(\xi,t)}{\partial t} + U_N\frac{\partial w_0(\xi,t)}{\partial \xi}\right\} e^{-i\frac{M\omega}{c_N\beta_N^2}(x-\xi)}$$

$$\times\ J_0\left\{\frac{\omega}{c_N\beta_N^2}\left[(x-\xi)^2 - \beta_N^2 y^2\right]^{1/2}\right\} d\xi \qquad \text{(A-23)}$$

If $\omega = 0$, we find from (A-23)

$$\phi_N = -\frac{1}{\beta_N}\int_0^{x-\beta_N y} U_N\frac{\partial w_0(\xi,t)}{\partial \xi}\, d\xi$$

$$= -\frac{U_N}{\beta_N}\left[w_0(x - \ _N y,t) - w_0(0,t)\right] \qquad \text{(A-24)}$$

APPENDIX B

LITERATURE SURVEY

The major purposes of this survey are to examine current literature pertaining to the unsteady flow associated with oscillating panels exposed to thick turbulent boundary layers with low supersonic Mach numbers and literature dealing with the obvious complicating mechanisms of unsteadiness, turbulence, compressibility, and transonic perturbations. Accordingly, the references are tabulated and briefly discussed in this order.

General Problem

The influence of the presence of the boundary layer on the flow past wavy walls is considered from the viewpoints of various authors in references [1 - 6]*. These papers are of prime importance in that they represent the various current approaches to approximation of the problem at hand. Miles [1] utilizes an inviscid, parallel shear flow model to investigate traveling waves; Benjamin [2] also considers traveling waves but with viscous effects included; and Mercer [3] considers standing waves with high frequency such as to omit consideration of a critical layer within the turbulent boundary layer. Fung [5] considers a somewhat crude model of the boundary layer represented as an inviscid subsonic uniform flow region beneath a uniform supersonic free stream, his primary objective being to include some effect of the boundary layer Mach number distribution. McClure [4,6] attempts to consider the general problem involving viscosity, compressibility, and turbulent boundary layer in a remarkable analysis. The complication of the theory, however, renders it difficult to utilize in a general investigation with a large number of parameters involved.

References [7] - [22] include work dealing with a number of related problems in unsteady boundary layer flows whereas references [23] - [32] pertain to turbulent boundary layers. References [33] - [39] include the effects of compressibility on the viscous flow and finally references [40] and [41] deal with the transonic flow past wave-shaped walls.

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APPENDIX C

MEAN FLOW EQUATIONS WITH SMALL PERTURBATIONS

The pressure distribution on a sinusoidally oscillating plate exposed to a thick turbulent boundary layer must be considered from the standpoint of the governing differential equations of continuity, momentum, and energy, together with an equation of state. This complex problem will first be considered on the form of a two-dimensional flow with emphasis on the resulting form of the continuity and momentum equations for small perturbations. The applicable governing equations are then:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{C-1}$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial p_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \tag{C-2}$$

$$\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = \frac{\partial p_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} \tag{C-3}$$

where the definitions employed are:

$$\left.\begin{aligned}
p &= -\frac{1}{3}(p_x + p_y + p_z) \\
p + p_x &= -\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x} \\
p + p_y &= -\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y} \\
p + p_z &= -\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\
\tau_{yx} &= \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned}\right\} \tag{C-4}$$

By using the continuity equation (C-1), the momentum equations (C-2) and (C-3) may be conveniently rewritten as

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) = \frac{\partial p_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \tag{C-5}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2) = \frac{\partial p_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} \tag{C-6}$$

The turbulent flow may then be represented as composed of "mean" and fluctuating components in all dependent variables where it is understood that the fluctuating components are random in nature and not to be confused with the regular fluctuations due to wall oscillations which are by definition periodic and part of the "mean" flow. The fluctuations due to turbulence are then defined by primed quantities as follows:

$$\left.\begin{aligned}
&u = \bar{u} + u' \;, \ldots \\
&\rho u = \overline{\rho u} + (\rho u)' \;, \ldots \\
&\rho = \bar{\rho} + \rho' \\
&p_i = \overline{p_i} + p_i' \quad ; \quad i = x,y,z \\
&\tau_{ij} = \overline{\tau_{ij}} + \tau_{ij}' \quad ; \quad i = x,y,z \quad ; \quad i \neq j
\end{aligned}\right\} \tag{C-7}$$

These definitions are then utilized in equations (C-1), (C-5), and (C-6), and the resulting equations time averaged over a period large in comparison to the time of random fluctuations but small in comparison to the time of "mean" variations such as the period of regular disturbances due to wall oscillations. The resulting "mean" flow equations are as given by Van Driest 23, Appendix B .

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x}\left(\overline{\rho u}\right) + \frac{\partial}{\partial y}\left(\overline{\rho v}\right) = 0 \qquad \text{(C-8)}$$

$$\bar{\rho}\,\frac{\partial \bar{u}}{\partial t} + \overline{\rho u}\,\frac{\partial \bar{u}}{\partial x} + \overline{\rho v}\,\frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial t}\left(-\,\overline{\rho' u'}\right) + \frac{\partial}{\partial x}\left[\bar{p}_x - \overline{(\rho u)' u'}\right]$$

$$+ \frac{\partial}{\partial y}\left[\bar{\tau}_{yx} - \overline{(\rho v)' u'}\right] \qquad \text{(C-9)}$$

$$\bar{\rho}\,\frac{\partial \bar{v}}{\partial t} + \overline{\rho u}\,\frac{\partial \bar{v}}{\partial x} + \overline{\rho v}\,\frac{\partial \bar{v}}{\partial y} = \frac{\partial}{\partial t}\left(-\,\overline{\rho' v'}\right) + \frac{\partial}{\partial y}\left[\bar{p}_y - \overline{(\rho v)' v'}\right]$$

$$+ \frac{\partial}{\partial x}\left[\bar{\tau}_{xy} - \overline{(\rho u)' v'}\right] \qquad \text{(C-10)}$$

where the barred quantities represent the "mean" values as given by the time average. If in addition to the shear stress, equation (C-4), the normal stresses are conventionally defined, one obtains:

$$\left.\begin{aligned}
\tau_{xx} &+ 2\mu\,\frac{\partial u}{\partial x} - \frac{2}{3}\,\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\
\tau_{yy} &= 2\mu\,\frac{\partial v}{\partial y} - \frac{2}{3}\,\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\
\tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned}\right\} \qquad \text{(C-11)}$$

These definitions allow equations (C-9) and (C-10) to be written in terms of the mean pressure, $\bar{p}$, by use of

$$\left.\begin{aligned} \bar{p}_x &= -\bar{p} + \overline{\tau_{xx}} \\ \bar{p}_y &= -\bar{p} + \overline{\tau_{yy}} \end{aligned}\right\} \qquad \text{(C-12)}$$

It is then noted that the "mean" values of the products of fluctuating quantities enter equations (C-9) and (C-10) as two alternations to the analogous laminar flow equations. The first of these is the appearance of the apparent stress terms $\overline{(\rho u)'u'}$, $\overline{(\rho v)'u'}$, $\overline{(\rho v)'v'}$, and $\overline{(\rho u)'v'}$; the second appears as the unsteadiness of the "mean" products $\overline{\rho'u'}$ and $\overline{\rho'v'}$.

This first alternation is handled in a more compact form by defining the "total" stresses as:

$$\left.\begin{aligned} \overline{\tau_{xx}} - \overline{(\rho u)'u'} &\equiv \overline{\tau_{xx}}^{t} \\ \overline{\tau_{yy}} - \overline{(\rho v)'v'} &\equiv \overline{\tau_{yy}}^{t} \\ \overline{\tau_{yx}} - \overline{(\rho v)'u'} &\equiv \overline{\tau_{yx}}^{t} \\ \overline{\tau_{xy}} - \overline{(\rho u)'v'} &\equiv \overline{\tau_{xy}}^{t} \neq \overline{\tau_{yx}}^{t} \end{aligned}\right\} \qquad \text{(C-13)}$$

The second alternation is, in effect, hidden by use of the identity

$$\frac{\partial}{\partial t}\left(\overline{\rho u_i}\right) = \frac{\partial}{\partial t}\left(\bar{\rho}\bar{u}_i\right) + \frac{\partial}{\partial t}\left(\overline{\rho' u_i'}\right) \qquad \text{(C-14)}$$

and the continuity equation (C-8) multiplied by the appropriate velocity component of the mean flow $\bar{u}$ or $\bar{v}$. With these improvements the system of equations governing the flow is:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x} (\overline{\rho u}) + \frac{\partial}{\partial y} (\overline{\rho v}) = 0 \tag{C-15}$$

$$\frac{\partial}{\partial t} (\overline{\rho u}) + \frac{\partial}{\partial x} (\overline{\rho u}\ \bar{u}) + \frac{\partial}{\partial y} (\overline{\rho v}\ \bar{u}) + \frac{\partial \bar{p}}{\partial x} = \frac{\partial}{\partial x} \overline{\tau_{xx}}^{t} + \frac{\partial}{\partial y} \overline{\tau_{yx}}^{t} \tag{C-16}$$

$$\frac{\partial}{\partial t} (\overline{\rho v}) + \frac{\partial}{\partial x} (\overline{\rho u}\ \bar{v}) + \frac{\partial}{\partial y} (\overline{\rho v}\ \bar{v}) + \frac{\partial \bar{p}}{\partial y} = \frac{\partial}{\partial y} \overline{\tau_{yy}}^{t} + \frac{\partial}{\partial x} \overline{\tau_{xy}}^{t} \tag{C-17}$$

These equations, as expressed, are in their least objectional form from the standpoint of the turbulent nature of the flow, in that the "mean" of products of the fluctuating quantities do not explicitly appear. Equations (C-16) and (C-17) are simply the Navier-Stokes equations for the "mean" flow written in the form of equations (C-5) and (C-6); as such they are not amenable to exact solution and recourse is made to approximate methods.

Small Perturbations

An obvious simplification to the preceeding general formulation may be accompolished by incorporating two reasonable assumptions:

1) The disturbances due to harmonic wall oscillations are assumed to be small. In effect, the flow is described as the sum of a "mean" steady flow plus a "mean" regular fluctuation which permits all flow variables to be defined as:

$$\underset{\text{"mean" flow}}{\overline{(\ )}} = \underset{\text{steady "mean" flow}}{\overline{(\ )}} + \underset{\text{unsteady regular fluctuation}}{\widetilde{(\ )}}$$

2) The variations of the steady "mean" flow components (the flow present in absence of wall oscillations) in the panel chordwise direction are neglected.

These assumptions are consistent with the approach used in inviscid flow over wavy walls of infinite extent wherein mathematical simplification is desirable and permissible. Further simplifications in the inviscid theories, by order of magnitude analysis, lead to the linearized theories for subsonic and supersonic flow and the nonlinear theory for transonic flow.

A detailed discussion of these assumptions and their consequences is given in the following development.

Steady Mean Turbulent Flow

The neglect of chordwise variations of flow variables in the "mean" steady flow component requires that

$$\frac{\partial (\ )}{\partial x} = 0 \tag{C-18}$$

In addition,

$$\frac{\partial (\ )}{\partial t} = 0$$

by definition and the governing equations (C-15) - (C-17) then reduce to the simple equations that follow:

$$\text{continuity} \qquad \frac{\partial}{\partial y} (\overline{\rho v}) = 0 \tag{C-19}$$

$$\text{x-momentum} \qquad \frac{\partial}{\partial y} (\overline{\rho v}\ \bar{u}) = \frac{\partial}{\partial y}\ \bar{\tau}_{yx}^{t} \tag{C-20}$$

$$\text{y-momentum} \qquad \frac{\partial}{\partial y} (\overline{\rho v}\ \bar{v}) + \frac{\partial \bar{p}}{\partial y} = \frac{\partial}{\partial y}\ \bar{\tau}_{yy}^{t} \tag{C-21}$$

A detailed examination of these equations yields some insight into the structure of the steady "mean" flow component, which is desirable, inasmuch as deviations from this state must be due to the wall oscillation. Consideration of each of these equations follows:

Continuity, Steady Mean Flow. As a result of equation (C-19) and the boundary condition

$$\overline{\rho v}\,\Big|_{wall} = 0$$

one obtains

$$\overline{\rho v} = 0 \tag{C-22}$$

Further, by definition

$$\overline{\rho v} = \bar{\rho}\bar{v} + \overline{\rho' v'}$$

so that

$$\bar{\rho}\bar{v} = - \overline{\rho' v'} \tag{C-23}$$

<u>x-Momentum, Steady Mean Flow</u>. By definition

$$\bar{\tau}_{yx}^{t} \equiv \bar{\tau}_{yx} - \overline{(\rho v)' u'}$$

$$\equiv \mu\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) - \overline{(\rho v)' u'}$$

so by virtue of equation (C-19) or (C-22) and the boundary condition

$$\overline{(\rho v)' u'}\Big|_{wall} = 0$$

equation (C-20) yields for the assumed steady "mean" flow:

$$\mu \frac{\partial \bar{u}}{\partial y} = \mu \frac{\partial \bar{u}}{\partial y}\Big|_{wall} + \overline{(\rho v)' u'} \tag{C-24}$$

<u>y-Momentum, Steady Mean Flow</u>. By definition

$$\bar{\tau}_{yy}^{t} = \bar{\tau}_{yy} - \overline{(\rho v)' v'}$$

$$= 2\mu \frac{\partial \bar{v}}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) - \overline{(\rho v)' v'}$$

so by virtue of equation (C-19) or (C-22) and the boundary condition

$$\overline{(\rho v)'v'}\Big|_{wall} = 0$$

equation (C-21) yields for the assumed steady "mean" flow:

$$\bar{p} = \bar{p}_{wall} + \frac{4}{3}\mu \left. \frac{\partial \bar{v}}{\partial y} - \frac{\partial \bar{v}}{\partial y} \right|_{wall} - \overline{(\rho v)'v'} \tag{C-25}$$

The resulting steady "mean" flow is then the compressible counterpart of a steady, parallel, incompressible shear flow. While a solution for the "mean" steady flow is not sought, it is both interesting and necessary to formulate the consequences of the neglect of chordwise variations of flow as given by equations (C-22), (C-23), (C-24), and (C-25). In particular, it is evident that a pressure variation through the thick turbulent boundary layer is present and contains, according to equation (C-25), contributions due the turbulent character of the flow and a combination viscous compressibility term. It is intuitively assumed that this pressure variation is small; in any case, it is the deviation from this state, due to the wall oscillations, in the presence of the boundary layer that is sought.

<u>Small Perturbation Equations</u>

The results of assuming small perturbations to the steady "mean" turbulent flow due to regular wall oscillations are found by rewriting the governing equations (C-15) - (C-17) replacing the total "mean" flow variables with their corresponding sum of steady and regular fluctuations. The deviations from the steady "mean" flow may then be examined in the form of governing equations for the perturbations inasmuch as the steady "mean" flow satisfies equations (C-19) - (C-21).

<u>Continuity, Perturbed Mean Flow</u>. Inasmuch as no products of variables appear in the continuity equation (C-15), the corresponding perturbation equation is given by

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial}{\partial x}(\widetilde{\rho u}) + \frac{\partial}{\partial y}(\widetilde{\rho v}) = 0 \tag{C-26}$$

x-Momentum, Perturbed Mean Flow. Equation (C-16) becomes upon substitution

$$\frac{\partial}{\partial t}(\overline{\rho u}) + \frac{\partial}{\partial t}(\widetilde{\rho u}) + \frac{\partial}{\partial x}\left[(\overline{\rho u} + \widetilde{\rho u})(\bar{u} + \tilde{u})\right] + \frac{\partial}{\partial y}\left[(\overline{\rho v} + \widetilde{\rho v})(\bar{u} + \tilde{u})\right]$$

$$+ \frac{\partial \bar{p}}{\partial x} + \frac{\partial \tilde{p}}{\partial x} = \frac{\partial}{\partial x}\left(\bar{\tau}_{xx}^{t} + \tilde{\tau}_{xx}^{t}\right) + \frac{\partial}{\partial y}\left(\bar{\tau}_{yx}^{t} + \tilde{\tau}_{yx}^{t}\right)$$

where the symbol ($\bar{}$) now refers only to the steady component of the "mean" flow. Neglecting products of small terms and utilizing equations (C-19) and (C-20) this equation reduces to the corresponding x-momentum perturbation equation:

$$\frac{\partial}{\partial t}(\widetilde{\rho u}) + \overline{\rho u}\,\frac{\partial \tilde{u}}{\partial x} + \bar{u}\,\frac{\partial \widetilde{\rho u}}{\partial x} + \widetilde{\rho v}\,\frac{\partial \bar{u}}{\partial y} + \bar{u}\,\frac{\partial \widetilde{\rho v}}{\partial y} + \frac{\partial \tilde{p}}{\partial x} = \frac{\partial}{\partial x}\tilde{\tau}_{xx}^{t} + \frac{\partial}{\partial y}\tilde{\tau}_{yx}^{t} \tag{C-27}$$

y-Momentum, Perturbed Mean Flow. Similarly equation (C-17) is expanded by substitution to obtain:

$$\frac{\partial}{\partial t}(\overline{\rho v}) + \frac{\partial}{\partial t}(\widetilde{\rho v}) + \frac{\partial}{\partial x}\left[(\overline{\rho u} + \widetilde{\rho u})(\bar{v} + \tilde{v})\right] + \frac{\partial}{\partial y}\left[(\overline{\rho v} + \widetilde{\rho v})(\bar{v} + \tilde{v})\right]$$

$$+ \frac{\partial \bar{p}}{\partial y} + \frac{\partial \tilde{p}}{\partial y} = \frac{\partial}{\partial y}\left(\bar{\tau}_{yy}^{t} + \tilde{\tau}_{yy}^{t}\right) + \frac{\partial}{\partial x}\left(\bar{\tau}_{xy}^{t} + \tilde{\tau}_{xy}^{t}\right)$$

and reduced in like manner to the y-momentum equation

$$\frac{\partial}{\partial t}(\widetilde{\rho v}) + \overline{\rho u}\,\frac{\partial \tilde{v}}{\partial x} + \bar{v}\,\frac{\partial \widetilde{\rho u}}{\partial x} + \widetilde{\rho v}\,\frac{\partial \bar{v}}{\partial y} + \bar{v}\,\frac{\partial \widetilde{\rho v}}{\partial y} + \frac{\partial \tilde{p}}{\partial y} = \frac{\partial}{\partial y}\tilde{\tau}_{yy}^{t} + \frac{\partial}{\partial x}\tilde{\tau}_{xy}^{t} \tag{C-28}$$

Perturbation Equations, Alternate Forms. Collecting the perturbation equations, one has

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial}{\partial x}(\widetilde{\rho u}) + \frac{\partial}{\partial y}(\widetilde{\rho v}) = 0 \tag{C-26}$$

$$\frac{\partial}{\partial t}(\widetilde{\rho u}) + \overline{\rho u}\,\frac{\partial \tilde{u}}{\partial x} + \bar{u}\,\frac{\partial \widetilde{\rho u}}{\partial x} + \widetilde{\rho v}\,\frac{\partial \bar{u}}{\partial y} + \bar{u}\,\frac{\partial \widetilde{\rho v}}{\partial y} + \frac{\partial \tilde{p}}{\partial x} = \frac{\partial}{\partial x}\tilde{\tau}_{xx}^{\;t} + \frac{\partial}{\partial y}\tilde{\tau}_{yx}^{\;t} \tag{C-27}$$

$$\frac{\partial}{\partial t}(\widetilde{\rho v}) + \overline{\rho u}\,\frac{\partial \tilde{v}}{\partial x} + \bar{v}\,\frac{\partial \widetilde{\rho u}}{\partial x} + \widetilde{\rho u}\,\frac{\partial \bar{v}}{\partial y} + \bar{v}\,\frac{\partial \widetilde{\rho v}}{\partial y} + \frac{\partial \tilde{p}}{\partial y} = \frac{\partial}{\partial y}\tilde{\tau}_{yy}^{\;t} + \frac{\partial}{\partial x}\tilde{\tau}_{xy}^{\;t} \tag{C-28}$$

Alternately, using equation (C-26), equations (C-27) and (C-28) may be rewritten as

$$\frac{\partial}{\partial t}(\widetilde{\rho u}) - \bar{u}\,\frac{\partial \tilde{\rho}}{\partial t} + \overline{\rho u}\,\frac{\partial \tilde{u}}{\partial x} + \widetilde{\rho v}\,\frac{\partial \bar{u}}{\partial y} + \frac{\partial \tilde{p}}{\partial x} = \frac{\partial}{\partial x}\tilde{\tau}_{xx}^{\;t} + \frac{\partial}{\partial y}\tilde{\tau}_{yx}^{\;t} \tag{C-27a}$$

$$\frac{\partial}{\partial t}(\widetilde{\rho v}) - \bar{v}\,\frac{\partial \tilde{\rho}}{\partial t} + \overline{\rho u}\,\frac{\partial \tilde{v}}{\partial x} + \widetilde{\rho v}\,\frac{\partial \bar{v}}{\partial y} + \frac{\partial \tilde{p}}{\partial y} = \frac{\partial}{\partial y}\tilde{\tau}_{yy}^{\;t} + \frac{\partial}{\partial x}\tilde{\tau}_{xy}^{\;t} \tag{C-28a}$$

Expanding the unsteady terms in equations (C-27a) and (C-28a) shows that

$$\frac{\partial}{\partial t}(\widetilde{\rho u}) - \bar{u}\,\frac{\partial \tilde{\rho}}{\partial t} = \bar{\rho}\,\frac{\partial \tilde{u}}{\partial t}$$

and

$$\frac{\partial}{\partial t}(\widetilde{\widetilde{\rho v}}) - \bar{v}\frac{\partial \tilde{\rho}}{\partial t} = \bar{\rho}\frac{\partial \tilde{v}}{\partial t}$$

and the momentum equations (C-27) and (C-28) are most conveniently written in the form

$$\bar{\rho}\frac{\partial \tilde{u}}{\partial t} + \overline{\rho u}\frac{\partial \tilde{u}}{\partial x} + \widetilde{\widetilde{\rho v}}\frac{\partial \bar{u}}{\partial y} + \frac{\partial \tilde{p}}{\partial x} = \frac{\partial}{\partial x}\tilde{\tau}_{xx}{}^{t} + \frac{\partial}{\partial y}\tilde{\tau}_{yx}{}^{t} \tag{C-27b}$$

$$\bar{\rho}\frac{\partial \tilde{v}}{\partial t} + \overline{\rho u}\frac{\partial \tilde{v}}{\partial x} + \widetilde{\widetilde{\rho v}}\frac{\partial \bar{v}}{\partial y} + \frac{\partial \tilde{p}}{\partial y} = \frac{\partial}{\partial y}\tilde{\tau}_{yy}{}^{t} + \frac{\partial}{\partial x}\tilde{\tau}_{xy}{}^{t} \tag{C-28b}$$

where, neglecting products of small terms

$$\widetilde{\widetilde{\rho v}} = \bar{\rho}\tilde{v} + \tilde{\rho}\bar{v} \tag{C-29}$$

Utilizing equation (C-29) and a similar result for $\widetilde{\widetilde{\rho u}}$, the convective derivatives in equation (C-26) may be expanded. The expansion of the latter yields

$$\frac{\partial}{\partial x}(\widetilde{\widetilde{\rho u}}) = \bar{\rho}\frac{\partial \tilde{u}}{\partial x} + \tilde{u}\frac{\partial \bar{\rho}}{\partial x} + \tilde{\rho}\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial \tilde{\rho}}{\partial x}$$

but the second and third terms on the right are zero by definition of the mean flow so that

$$\frac{\partial}{\partial x}(\widetilde{\widetilde{\rho u}}) = \bar{\rho}\frac{\partial \tilde{u}}{\partial x} + \bar{u}\frac{\partial \tilde{\rho}}{\partial x} \tag{C-30}$$

and likewise

$$\frac{\partial}{\partial y}(\widetilde{\widetilde{\rho v}}) = \bar{\rho}\frac{\partial \tilde{v}}{\partial y} + \tilde{v}\frac{\partial \bar{\rho}}{\partial y} + \tilde{\rho}\frac{\partial \bar{v}}{\partial y} + \bar{v}\frac{\partial \tilde{\rho}}{\partial y} \tag{C-31}$$

The continuity equation (C-26) may then be written after substitution of equations (C-30) and (C-31) as

$$\frac{\partial \tilde{\rho}}{\partial t} + \bar{u}\,\frac{\partial \tilde{\rho}}{\partial x} + \bar{v}\,\frac{\partial \tilde{\rho}}{\partial y} + \bar{\rho}\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) + \tilde{v}\,\frac{\partial \bar{\rho}}{\partial y} + \tilde{\rho}\,\frac{\partial \bar{v}}{\partial y} = 0 \qquad \text{(C-32)}$$

<u>Perturbation Equations, Final Form</u>. Finally, collecting results, equations (C-32), (C-27b), and (C-28b) provide the governing equations of momentum and continuity in their most tractable form:

$$\frac{\partial \tilde{\rho}}{\partial t} + \bar{u}\,\frac{\partial \tilde{\rho}}{\partial x} + \bar{v}\,\frac{\partial \tilde{\rho}}{\partial y} + \bar{\rho}\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) + \tilde{v}\,\frac{\partial \bar{\rho}}{\partial y} + \tilde{\rho}\,\frac{\partial \bar{v}}{\partial y} = 0 \qquad \text{(C-33a)}$$

$$\bar{\rho}\,\frac{\partial \tilde{u}}{\partial t} + \overline{\rho u}\,\frac{\partial \tilde{u}}{\partial x} + \widetilde{\rho v}\,\frac{\partial \bar{u}}{\partial y} + \frac{\partial \tilde{p}}{\partial x} = \frac{\partial}{\partial x}\,\tilde{\tau}_{xx}{}^{t} + \frac{\partial}{\partial y}\,\tilde{\tau}_{yx}{}^{t} \qquad \text{(C-33b)}$$

$$\bar{\rho}\,\frac{\partial \tilde{v}}{\partial t} + \overline{\rho u}\,\frac{\partial \tilde{v}}{\partial x} + \widetilde{\rho v}\,\frac{\partial \bar{v}}{\partial y} + \frac{\partial \tilde{p}}{\partial y} = \frac{\partial}{\partial y}\,\tilde{\tau}_{yy}{}^{t} + \frac{\partial}{\partial x}\,\tilde{\tau}_{xy}{}^{t} \qquad \text{(C-33c)}$$

with

$$\widetilde{\rho v} = \bar{\rho}\tilde{v} + \tilde{\rho}\bar{v} \qquad \text{(C-33d)}$$

and

$$\left.\begin{aligned} \tilde{\tau}_{xx}{}^{t} &= 2\mu\,\frac{\partial \tilde{u}}{\partial x} - \frac{2}{3}\,\mu\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) \\ \tilde{\tau}_{yx}{}^{t} &= \tilde{\tau}_{xy}{}^{t} = \mu\left(\frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x}\right) \\ \tilde{\tau}_{yy}{}^{t} &= 2\mu\,\frac{\partial \tilde{v}}{\partial y} - \frac{2}{3}\,\mu\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) \end{aligned}\right\} \qquad \text{(C-33e)}$$

In formulating these perturbation equations, it has been assumed, as evidenced by equations (C-33e), that the perturbations do not influence the character of the turbulence, that is:

$$\widetilde{(\rho u)' u'} = 0$$

$$\widetilde{(\rho v)' u'} = 0$$

$$\widetilde{(\rho v)' v'} = 0$$

$$\widetilde{(\rho u)' v'} = 0$$

This assumption effectively eliminates the indeterminate situation due to the presence of turbulence and its corresponding "apparent" stress terms and allows one to consider the present problem as a pseudo-laminar flow. Examination of the governing equations, on the other hand, reveals that the mathematical situation consists of three equations in which the "mean" steady flow properties

$$\bar{\rho},\ \bar{u},\ \bar{v},\ \overline{\rho u},\ \mu$$

are regarded as known or prescribed coefficients and the perturbation quantities

$$\tilde{\rho},\ \tilde{u},\ \tilde{v},\ \tilde{p}$$

are regarded as the dependent variables. Obviously, except in the case of incompressible perturbations, this situation necessitates consideration of the energy equation and an equation of state.

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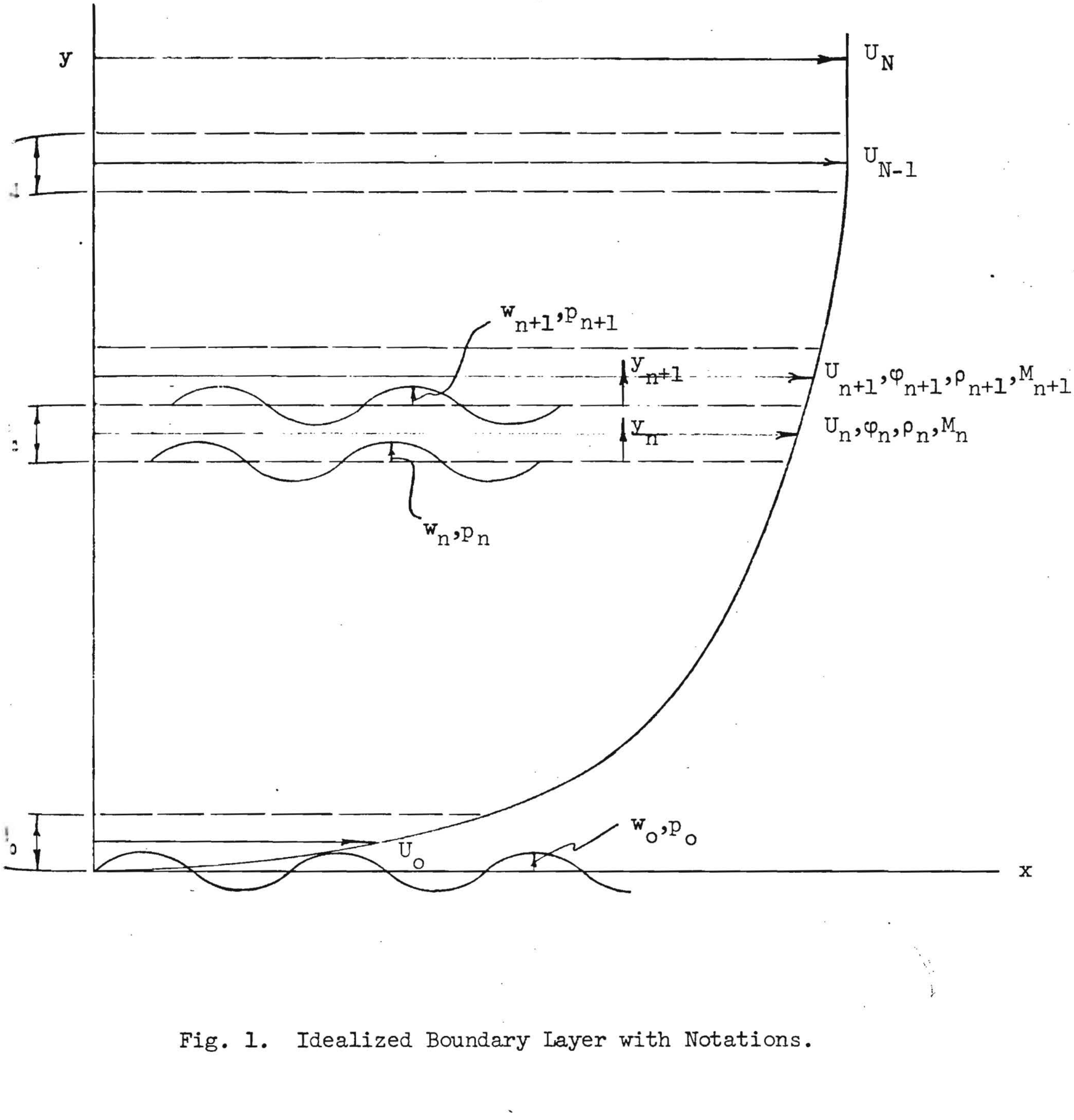

Fig. 1. Idealized Boundary Layer with Notations.